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On a Gauss-Weierstrass generalized integral

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. In the present note we consider the integral

$$W_\nu(f;r,t) = \int_0^\infty K_\nu(r,s,t)f(s)\,ds,$$

where $r > 0$, $t > 0$, $\nu \geqslant -\frac{1}{2}$,

$$K_\nu(r,s,t) = \frac{1}{2t}r^{-\nu}s^{\nu+1}I_\nu\left(\frac{rs}{2t}\right)\exp\left(-\frac{r^2+s^2}{4t}\right),$$

I_ν is a modified Bessel function. The integral $W_\nu(f;r,t)$ will be called a Gauss-Weierstrass generalized integral. We investigate, among others, asymptotic properties of this integral in the space $E_K = \{f : [0,+\infty) \to \mathbb{R} :\ f$ is continuous on $[0,+\infty)$ and $f(r) = O(e^{Kr^2})$ as $r \to +\infty\}$; $K \geqslant 0$.

1. Introduction

Let $C(\mathbb{R}_+)$ be the set of all continuous real-valued functions on $\mathbb{R}_+ = [0,+\infty)$. Let $K \geqslant 0$. We consider the space E_K defined by

$$E_K = \left\{f \in C(\mathbb{R}_+) :\ \exists\, M > 0\ \forall\, r \in \mathbb{R}_+\ \ |f(r)| \leqslant Me^{Kr^2}\right\}.$$

Let

$$\|f\|_K = \sup_{\mathbb{R}_+}|f(r)|e^{-Kr^2}.$$

In the space E_K we consider the operator $W_\nu(f)$ defined by

$$W_\nu(f)(r,t) = W_\nu(f;r,t) = \int_0^\infty K_\nu(r,s,t)f(s)\,ds \tag{1}$$

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where $\nu \geqslant -\frac{1}{2}$, $r > 0$, $t > 0$, $K_\nu(r,s,t) = \frac{1}{2t} r^{-\nu} s^{\nu+1} I_\nu\left(\frac{rs}{2t}\right) \exp\left(-\frac{r^2+s^2}{4t}\right)$, and I_ν is a modified Bessel function ([2]);

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{\nu+2k}}{2^{\nu+2k} k! \Gamma(\nu+k+1)}. \tag{2}$$

For $\nu = -\frac{1}{2}$ we get $I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \operatorname{ch} z$ and

$$W_{-\frac{1}{2}}(f;r,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \tilde{f}(s) \exp\left(-\frac{(r-s)^2}{4t}\right) ds, \tag{3}$$

where

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \geqslant 0, \\ f(-s) & \text{if } s < 0. \end{cases}$$

The integral (3) is the classical Gauss-Weierstrass integral and a solution of the heat equation.

In the general case, i.e. if $\nu \geqslant -\frac{1}{2}$, the function $W_\nu(f)$ is an example of the solution of the generalized heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2\nu+1}{r} \frac{\partial u}{\partial r}. \tag{4}$$

If $\nu = \frac{n}{2} - 1$, $n = 1, 2, \dots$, the equation (4) is the heat equation (in $\mathbb{R}_{n+1}$) in the radial coordinates and the function $W_\nu(f)$ is a solution of this equation.

If $f(s) = s^{2k}$, $k \in \mathbb{N}$, the function $W_\nu(f)$ is a polynomial. The polynomials determined in this way are called radial heat polynomials ([1]). In the paper [1] the expansions of solutions of the equation (4) in terms of radial heat polynomials and their Appell transforms (associated functions) have been studied.

The integral (1) can be considered as the example of the generalized convolution. Indeed, consider in $\mathbb{R}^n$ the convolution

$$(F * G)(x) = \int_{\mathbb{R}^n} G(x-y) F(y)\, dy, \quad x \in \mathbb{R}^n,$$

of two radial functions. If $F(x) = f(|x|)$, $G(x) = g(|x|)$, $|x|^2 = \sum_{i=1}^n x_i^2$, then

$$(F * G)(x) = (f * g)(|x|) = \int_{\mathbb{R}^n} f(|x-y|) g(|y|)\, dy.$$

In the spherical coordinates we get

$$(f * g)(|x|)$$
$$= \frac{2\pi^{\nu+\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty f(s) s^{2\nu+1} \int_0^\pi g\left(\left(|x|^2 + s^2 - 2|x| s \cos\theta\right)^{\frac{1}{2}}\right) \sin^{2\nu}\theta \, d\theta\, ds$$

where $\nu = \frac{n}{2} - 1$. By allowing n to be an arbitrary real number greater than 1 we obtain the operator $\otimes$ defined by

$$(f \otimes g)(r) = \frac{2\pi^{\nu+\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty f(s)s^{2\nu+1} \int_0^\pi g\left(\left(r^2+s^2-2rs\cos\theta\right)^{\frac{1}{2}}\right) \sin^{2\nu}\theta \, d\theta \, ds,$$

$r > 0$, $\nu \geqslant -\frac{1}{2}$. This operator is a generalized convolution ([4]). Let

$$g_t(\varrho) = (4\pi t)^{-\nu-1} \exp\left(-\frac{\varrho^2}{4t}\right), \quad t > 0, \ \varrho \in \mathbb{R}.$$

Using the properties of the modified Bessel function ([2]) we get

$$\int_0^\pi g_t\left(\left(r^2+s^2-2rs\cos\theta\right)^{\frac{1}{2}}\right) \sin^{2\nu}\theta \, d\theta = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{4t\pi^{\nu+\frac{1}{2}}} (sr)^{-\nu} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\nu\left(\frac{rs}{2t}\right).$$

Hence $W_\nu(f; r, t) = (f \otimes g_t)(r)$.

In the present paper in Section 2 we will prove the following theorems:

THEOREM 1

Let $f \in E_K$. The function $W_\nu(f)$ is of the class C^∞ in the set $D = \{(r,t) : r > 0, \ 0 < t < \frac{1}{4K}\}$ (if $K = 0$, then $0 < t < +\infty$). Moreover, the function $W_\nu(f)$ is a solution of equation (4) *in D.*

THEOREM 2

Let $f \in E_K$. If $K > 0$, then for each $\delta > 0$ and $t \in \left(0, \frac{\delta}{4K(K+\delta)}\right)$ the function

$$W_\nu(f; \cdot, t) : r \longrightarrow W_\nu(f; r, t) \tag{5}$$

is in $E_{K+\delta}$ and

$$\|W_\nu(f; \cdot, t)\|_{K+\delta} \leqslant \left(1 + \frac{\delta}{K}\right)^{\nu+1} \|f\|_K .$$

If $K = 0$, then the function (5) *is in E_K and*

$$\|W_\nu(f; \cdot, t)\|_K \leqslant \|f\|_K,$$

THEOREM 3

Let $f \in E_K$. For every $r_o \in [0, +\infty)$

$$\lim_{(r,t)\to(r_o,0^+)} W_\nu(f; r, t) = f(r_o).$$

THEOREM 4

Let $f \in E_K$. *For every* $0 < \alpha < \beta$

$$\lim_{t\to 0+} W_\nu(f;r,t) = f(r) \tag{6}$$

uniformly on $[\alpha,\beta]$.

In Section 3 we will state, using the moduli of continuity, some estimates of the rate of convergence (6).

2. Proofs

In this section we will prove Theorems 1, 2, 3 and 4.

Proof of Theorem 1. Let $a, b, A, B \in \mathbb{R}$ and $0 < a < B$, $0 < b < B < \frac{1}{4K}$. Consider the set

$$D(a,b,A,B) = \{(r,t) :\ a \leqslant r \leqslant A,\ b \leqslant t \leqslant B\}.$$

Let $p, q \in \mathbb{N}$. Applying the induction and the properties of the modified Bessel function ([2]) we note that the integral

$$\int_0^\infty \frac{\partial^{p+q}}{\partial r^p \partial t^q} K_\nu(r,s,t) f(s)\, ds \tag{7}$$

is a linear combination of the integrals of the form

$$\frac{1}{(2t)^k} r^{-\mu+\lambda} \int_0^\infty s^{\mu+l} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\mu\left(\frac{rs}{2t}\right) f(s)\, ds,$$

where $k \geqslant 1$; $\lambda \geqslant 0$; $\mu \geqslant \nu$; $l \geqslant 1$; $k, \mu, \lambda, l \in \mathbb{N}$.

We get

$$\left|\frac{1}{(2t)^k} r^{-\mu+\lambda} s^{\mu+l} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\mu\left(\frac{rs}{2t}\right) f(s)\right| \leqslant C_1 \|f\|_K s^{\mu+l} \exp\left(-\frac{s^2}{4B} + Ks^2\right) I_\mu\left(\frac{rs}{2t}\right)$$

for $(r,t) \in D(a,b,A,B)$, $s > 0$, where C_1 is a positive constant.

Remark that

$$4^k k!\Gamma(\nu+k+1) \geqslant (2k)!\Gamma(\nu+1)$$

for every $k \in \mathbb{N}$ and $\nu \geqslant -\frac{1}{2}$. Hence, by (2), we get

$$|I_\nu(z)| \leqslant \left(\frac{1}{2}z\right)^\nu \frac{1}{\Gamma(\nu+1)} \exp z \tag{8}$$

for $z > 0$ and $\nu \geqslant -\frac{1}{2}$.

This implies that

$$\begin{aligned}\left|\frac{1}{(2t)^k} r^{-\mu+\lambda} s^{\mu+l} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\mu\left(\frac{rs}{2t}\right) f(s)\right| \\ \leqslant C_2\|f\|_K\, s^{2\mu+l} \exp\left(-\frac{1-4KB}{4B}s^2+\frac{A}{2b}s\right)\end{aligned} \tag{9}$$

for $(r,t) \in D(a,b,A,B), s > 0$, where C_2 is a positive constant. Observe that the integral

$$\int_0^\infty s^{2\mu+l} \exp\left(-\frac{1-4KB}{4B}s^2+\frac{A}{2b}s\right) ds$$

is convergent, hence by (9), the integral

$$\int_0^\infty s^{\mu+l} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\mu\left(\frac{rs}{2t}\right) f(s)\, ds$$

and the integral (7) are uniformly convergent on $D(a,b,A,B)$.

This implies that

$$\frac{\partial^{p+q}}{\partial r^p \partial t^q}\int_0^\infty K_\nu(r,s,t) f(s)\, ds = \int_0^\infty \frac{\partial^{p+q}}{\partial r^p \partial t^q} K_\nu(r,s,t) f(s)\, ds$$

in D. Consequently, the function $W_\nu(f)$ is of class C^∞ in D. It is easy to verify that

$$\frac{\partial K_\nu(r,s,t)}{\partial t} = \frac{\partial^2 K_\nu(r,s,t)}{\partial r^2} + \frac{2\nu+1}{r}\,\frac{\partial K_\nu(r,s,t)}{\partial r}$$

in D for every $s \in (0,+\infty)$.

This completes the proof of Theorem 1.

Proof of Theorem 2. By (2) we obtain

$$\int_0^\infty s^{\mu+1} \exp\left(-\alpha s^2\right) I_\mu(\beta s)\, ds = \frac{\beta^\mu}{(2\alpha)^{\mu+1}} \exp\left(\frac{\beta^2}{4a}\right) \tag{10}$$

for $\alpha > 0$, $\beta > 0$, $\mu > -1$.

This implies that

$$\begin{aligned}&|W_\nu(f;r,t)| \\ &\leqslant \|f\|_K \frac{r^{-\nu}}{2t} \exp\left(-\frac{r^2}{4t}\right) \int_0^\infty s^{\nu+1} \exp\left(-\left(\frac{1}{4t}-K\right)s^2\right) I_\nu\left(\frac{rs}{2t}\right) ds \\ &= \|f\|_K (1-4Kt)^{-\nu-1} \exp\left(\frac{Kr^2}{1-4Kt}\right)\end{aligned}$$

for $(r,t) \in D$.

If $K = 0$ we get

$$\|W_\nu(f;\cdot,t)\|_K \leqslant \|f\|_K.$$

Suppose that $K > 0$. Let $\delta > 0$ and $t \in \left(0, \frac{\delta}{4K(K+\delta)}\right)$. Hence $\frac{K}{1-4Kt} < K + \delta$ and

$$\|W_\nu(f;\cdot,t)\|_{K+\delta} \leqslant \|f\|_K(1-4Kt)^{-\nu-1} \leqslant \|f\|_K \left(1 + \frac{\delta}{K}\right)^{\nu+1}.$$

The proof of Theorem 2 is completed.

Remark 1

The assumption that $f \in E_K$ $(K > 0)$ does not imply that the function (5) is in E_K. Indeed, if $h(r) = \exp r^2$ then $h \in E_1$, and by the formula (10) we obtain

$$W_\nu(h;r,t) = (1-4t)^{-\nu-1} \exp\left(\frac{r^2}{1-4t}\right)$$

for $r > 0$ and $0 < t < \frac{1}{4}$. From the above it follows that $W_\nu(h;r,t)\exp\left(-r^2\right)$ is not bounded in $[0,+\infty)$ for each $t \in \left(0, \frac{1}{4}\right)$.

To prove Theorem 3 we shall need the following lemmas.

Lemma 1

$$\int_0^\infty K_\nu(r,s,t)\,ds = 1$$

for every $r > 0$ and $t > 0$.

Proof. From (2) and from the inequality

$$s^\alpha e^{-s} \leqslant \alpha^\alpha e^{-\alpha}, \quad s > 0,\ \alpha > 0$$

it follows that the series

$$\sum_{k=0}^\infty \frac{r^{2k} s^{2\nu+2k+1} \exp\left(-\frac{s^2}{4t}\right)}{(4t)^{\nu+2k+1} k!\Gamma(\nu+k+1)}$$

is uniformly convergent on $[0,+\infty)$ with respect to s for every $r > 0$, $t > 0$, $\nu \geqslant -\frac{1}{2}$.

Hence

$$\int_0^\infty K_\nu(r,s,t)\,ds$$

$$= \frac{1}{2t}\exp\left(-\frac{r^2}{4t}\right)\sum_{k=0}^\infty \frac{r^{2k}}{(4t)^{2k+\nu} k!\Gamma(\nu+k+1)} \int_0^\infty s^{2\nu+2k+1}\exp\left(-\frac{s^2}{4t}\right)ds$$

$$= \frac{1}{4t}\exp\left(-\frac{r^2}{4t}\right)\sum_{k=0}^{\infty}\frac{r^{2k}(4t)^{2\nu+k+1}\Gamma(\nu+k+1)}{(4t)^{2k+\nu}k!\Gamma(\nu+k+1)}$$

$$= \exp\left(-\frac{r^2}{4t}\right)\sum_{k=0}^{\infty}\frac{r^{2k}}{(4t)^k k!} = 1.$$

LEMMA 2

Let $\beta > \alpha > 0$. If $\delta \in (0,\alpha)$, then

$$\lim_{t\to 0^+}\int_0^{r-\delta} K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds = 0,$$

uniformly on $[\alpha,\beta]$.

Proof. From inequality (8) it follows that

$$\int_0^{r-\delta} K_\nu(r,s,t)\exp\left(Ks^2\right)ds$$

$$\leqslant C_1 t^{-\nu-1}\int_0^{r-\delta} s^{2\nu+1}\exp\left(-\frac{(r-s)^2}{4t}+Ks^2\right)ds$$

$$\leqslant C_2 t^{-\nu-1}\exp\left(-\frac{\delta^2}{4t}\right)$$

for every $r \in [\alpha,\beta]$, where C_1, C_2 are positive constants independent of r.

Hence

$$\lim_{t\to 0^+}\int_0^{r-\delta} K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds = 0,$$

uniformly on $[\alpha,\beta]$.

LEMMA 3

Let $\delta > 0$, $\beta > 0$. Then

$$\lim_{t\to 0^+}\int_{r+\delta}^{\infty} K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds = 0,$$

uniformly on $(0,\beta]$.

Proof. Similarly as in the proof of Lemma 2 we get

$$\int_{r+\delta}^{+\infty} K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds$$

$$\leqslant C_1 t^{-\nu-1}\int_{r+\delta}^{+\infty} s^{2\nu+1}\exp\left(Ks^2-\frac{(r-s)^2}{4t}\right)ds$$

where C_1 is a positive constant.

Observe that

$$Ks^2 - \frac{(r-s)^2}{4t} = \frac{Kr^2}{1-4Kt} - \frac{1-4Kt}{4t}\left(s - \frac{r}{1-4Kt}\right)^2.$$

Thus

$$\int_{r+\delta}^{+\infty} K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds$$
$$\leqslant C_1 t^{-\nu-1}\exp\frac{K\beta^2}{1-4Kt}\int_{r+\delta}^{+\infty} s^{2\nu+1}\exp\left(-\frac{1-4Kt}{4t}\left(s-\frac{r}{1-4Kt}\right)^2\right)ds$$

for $0 < r \leqslant \beta$. Let $0 < \alpha < \frac{\delta}{\beta+\delta}$, then $\alpha < 1$ and

$$0 < \frac{\delta - \alpha(\beta+\delta)}{(1-\alpha)(\beta+\delta)} < 1.$$

Assume that $K > 0$. Let $0 < t < \frac{1}{4K}\eta$, where

$$\eta = \frac{\delta - \alpha(\beta+\delta)}{(1-\alpha)(\beta+\delta)}.$$

It follows that $0 < t < \frac{1}{4K}$ and

$$\frac{4Kt + (1-4Kt)\alpha}{1-4Kt}\, r < (1-\alpha)\delta \tag{11}$$

for $0 < r \leqslant \beta$.

From (11) we obtain

$$(1-\alpha)(r+\delta) \geqslant \frac{r}{1-4Kt}$$

and

$$\left(s - \frac{r}{1-4Kt}\right) \geqslant s\alpha$$

for $s > r + \delta$ and $0 < r \leqslant \beta$.

Hence

$$\int_{r+\delta}^{+\infty} K_\nu(r,s,t)\exp\left(Ks^2\right)ds$$
$$\leqslant C_1 t^{-\nu-1}\exp\frac{K\beta^2}{1-\eta}\int_{r+\delta}^{+\infty} s^{2\nu+1}\exp\left(-\frac{1-4Kt}{4t}s^2\alpha^2\right)ds$$
$$\leqslant C_2 t^{-\nu-1}\int_{\delta}^{+\infty} s^{2\nu+1}\exp\left(-\frac{1-4Kt}{4t}\alpha^2 s^2\right)ds$$

$$= C_2 t^{-\nu-1} \left(\frac{4t}{1-4Kt} \right)^{\nu+1} \int_{\sqrt{\frac{1-4Kt}{4t}}\alpha\delta}^{+\infty} s^{2\nu+1} \exp\left(-s^2\right) ds$$

$$\leqslant C_3 \int_{\frac{\gamma}{2\sqrt{t}}}^{+\infty} s^{2\nu+1} \exp\left(-s^2\right) ds$$

for every $0 < r \leqslant \beta$, where $\gamma = \sqrt{1-\eta}\,\alpha\delta$, C_2, C_3 are positive constant.

The integral

$$\int_0^{\infty} s^{2\nu+1} \exp\left(-s^2\right) ds$$

is convergent for $\nu \geqslant -\frac{1}{2}$. There exist $N > 0$ such that

$$\int_N^{+\infty} s^{2\nu+1} \exp\left(-s^2\right) ds < \frac{\varepsilon}{C_3}.$$

Hence

$$\int_{r+\delta}^{+\infty} K_\nu(r,s,t) \exp\left(Ks^2\right) ds \leqslant \varepsilon$$

for $0 < r \leqslant \beta$ and $0 < t < \min\left(\frac{\eta}{4K}, \frac{\gamma^2}{4N^2}\right)$.

If $K = 0$, similarly we get

$$\int_{r+\delta}^{+\infty} K_\nu(r,s,t)\, ds < \varepsilon$$

for $0 < r \leqslant \beta$ and $0 < t < \frac{\gamma^2}{4N^2}$.

This completes the proof of Lemma 3.

Proof of Theorem 3. Let $r_0 \geqslant 0$. By Lemma 1 we get

$$|W_\nu(f;r,t) - f(r_0)| \leqslant \int_0^{\infty} K_\nu(r,s,t) |f(s) - f(r_0)|\, ds.$$

Let $\varepsilon > 0$. By assumption there exists $\delta > 0$ such that

$$|f(s) - f(r_0)| < \frac{\varepsilon}{2} \tag{12}$$

for $|s - r_0| < \delta$ if $r_0 > 0$ and for $s \in (0,\delta)$ if $r_0 = 0$.

First we assume that $r_0 = 0$. We get

$$\begin{aligned}\int_0^{\infty} K_\nu(r,s,t) |f(s) - f(r_0)|\, ds &\leqslant \int_0^{\delta} K_\nu(r,s,t) |f(s) - f(r_0)|\, ds \\ &\quad + \int_\delta^{\infty} K_\nu(r,s,t) |f(s) - f(r_0)|\, ds \\ &= A + B\end{aligned}$$

By (12) and Lemma 1 we obtain

$$A \leqslant \frac{\varepsilon}{2}\int_0^\delta K_\nu(r,s,t)\,ds \leqslant \frac{\varepsilon}{2}\int_0^\infty K_\nu(r,s,t)\,ds = \frac{\varepsilon}{2},$$

for every $r > 0$.

If $0 < r < \frac{\delta}{2}$, then $r + \frac{\delta}{2} < \delta$ and

$$\begin{aligned} B &= \int_\delta^\infty K_\nu(r,s,t)\,|f(s) - f(r_0)|\,ds \\ &\leqslant 2\|f\|_K \int_\delta^\infty K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds \\ &\leqslant 2\|f\|_K \int_{r+\frac{\delta}{2}}^{+\infty} K_\nu(r,s,t)\exp\left(Ks^2\right)\,ds. \end{aligned}$$

The above inequality and Lemma 3 imply that there exists $\eta > 0$ such that $B < \frac{\varepsilon}{2}$ for $0 < t < \eta$, $0 < r < \frac{\delta}{2}$. Hence

$$|W_\nu(f;r,t) - f(0)| < \varepsilon$$

for $0 < t < \eta, 0 < r < \frac{\delta}{2}$. It follows that $\lim_{(r,t)\to(0^+,0^+)} W_\nu(f;r,t) = f(0)$.

Let $r_0 > 0$. We get

$$\begin{aligned} |W_\nu(f;r,t) - f(r_0)| \leqslant &\int_0^{r_0-\delta} K_\nu(r,s,t)\,|f(s) - f(r_0)|\,ds \\ &+ \int_{|s-r_0|<\delta} K_\nu(r,s,t)\,|f(s) - f(r_0)|\,ds \\ &+ \int_{r_0+\delta}^{+\infty} K_\nu(r,s,t)\,|f(s) - f(r_0)|\,ds \\ = &\; A + B + C. \end{aligned}$$

By (12) and Lemma 1 we get $B < \frac{\varepsilon}{2}$ for every $r > 0$. Let $|r - r_0| < \frac{\delta}{2}$, then $r_0 - \delta < r - \frac{\delta}{2}$ and $r + \frac{\delta}{2} < r_0 + \delta$.

Hence

$$\begin{aligned} &A + C \\ &\leqslant 2\|f\|_K \left(\int_0^{r-\frac{\delta}{2}} K_\nu(r,s,t)\exp\left(Ks^2\right)ds + \int_{r+\frac{\delta}{2}}^{+\infty} K_\nu(r,s,t)\exp\left(Ks^2\right)ds\right). \end{aligned}$$

Lemma 2 and 3 imply that there exists $\eta > 0$ such that $A + C < \frac{\varepsilon}{2}$ for $0 < t < \eta$ and $|r - r_0| < \frac{\delta}{2}$. Finally

$$|W_\nu(f;r,t) - f(r_0)| < \varepsilon$$

for $0 < t < \eta$ and $|r - r_0| < \frac{\delta}{2}$. This ends the proof of Theorem 3.

The proof of Theorem 4 is similar to that of Theorem 3.

3. Rate of convergence

In this section we will state some estimates of the rate of convergence of the integral $W_\nu(f;r,t)$ as $t \longrightarrow 0^+$. We shall apply the method used in the paper [3].

Using (2) we get

$$\int_0^\infty s^{\nu+2\mu+1} \exp\left(-as^2\right) I_\nu(\beta s)\, ds = \sum_{k=0}^{\infty} \frac{\beta^{\nu+2k}\Gamma(\nu+k+\mu+1)}{k!\Gamma(\nu+k+1)a^{\nu+k+\mu+1}2^{\nu+2k+1}}.$$

for $\nu \geqslant -\frac{1}{2}$, $\mu \geqslant 0$, $a > 0$, $\beta > 0$.

Hence

$$W_\nu(f_i;r,t) = \begin{cases} 1 & \text{for } i = 0, \\ r^2 + 4t(\nu+1) & \text{for } i = 1, \\ r^4 + 8tr^2(\nu+2) + 16t^2(\nu+1)(\nu+2) & \text{for } i = 2, \end{cases} \tag{13}$$

where $f_i(s) = s^{2i}$.

Observe that

$$W_\nu(f;r,t) = \frac{1}{4t} r^{-\nu} \exp\left(-\frac{r^2}{4t}\right) \int_0^\infty s^{\frac{\nu}{2}} \exp\left(-\frac{s}{4t}\right) I_\nu\left(\frac{r\sqrt{s}}{2t}\right) f(\sqrt{s})\, ds.$$

Putting

$$V_\nu(g;x,t) = \frac{1}{4t} \int_0^\infty \left(\frac{s}{x}\right)^{\frac{\nu}{2}} \exp\left(-\frac{x+s}{4t}\right) I_\nu\left(\frac{\sqrt{xs}}{2t}\right) g(s)\, ds$$

we obtain

$$W_\nu(f;r,t) = V_\nu\left(g;r^2,t\right),$$

where $f(s) = g(s^2)$.

By (13) we get

$$V_\nu(g_i;x,t) = \begin{cases} 1 & \text{for } i = 0, \\ x + 4t(\nu+1) & \text{for } i = 1, \\ x^2 + 8tx(\nu+2) + 16t^2(\nu+1)(\nu+2) & \text{for } i = 2. \end{cases} \tag{14}$$

$$V_\nu\left(\varphi_x;x,t\right) = 8tx + 16t^2(\nu+1)(\nu+2) \tag{15}$$

where $g_i(s) = s^i$, $\varphi_x(s) = (s-x)^2$.

We shall prove the following

LEMMA 4

Let $g \in C(\mathbb{R}_+)$. Assume that

$$\omega(g,h) = \sup_{\substack{0 \leqslant t \leqslant h \\ x > 0}} |g(x+h) - g(x)| < +\infty$$

for every $h > 0$. We have

$$|V_\nu(g;x,t) - g(x)| \leqslant L\,\omega\Big(g; \sqrt{8tx + 16t(\nu+1)(\nu+2)}\Big) \tag{16}$$

for $x > 0$ and $t > 0$, where L is a positive constant.

Proof. First we suppose that g is continuously differentable on $[0,+\infty)$ and g' is bounded on $[0,+\infty)$. We have

$$g(s) = g(x) + \int_x^s g'(\tau)\,d\tau.$$

Hence, from Lemma 1, formula (15) and from the Hölder inequality we obtain

$$\begin{aligned} |V_\nu(g;x,t) - g(x)| &\leqslant \sup_{[0,+\infty)} |g'(s)| V_\nu\left(\Psi_x, x, t\right) \\ &\leqslant \sup_{[0,+\infty)} |g'(s)| \left(V_\nu\left(\varphi_x; x,t\right)\right)^{\frac{1}{2}} \left(V_\nu\left(g_0; x,t\right)\right)^{\frac{1}{2}} \\ &= \sup_{[0,+\infty)} |g'(s)| \sqrt{8tx + 16t^2(\nu+1)(\nu+2)} \end{aligned}$$

for $x > 0$, $t > 0$, where $\Psi_x(s) = |s - x|$.

Let $g \in C(\mathbb{R}_+)$. Assume that $\omega(g,h) < +\infty$ for every $h > 0$. Let g_h be the Steklov mean of g, i.e.

$$g_h(x) = \frac{1}{h}\int_0^h g(x+\tau)\,d\tau, \quad x > 0,\ h > 0.$$

We have

$$g(x) - g_h(x) = \frac{1}{h}\int_0^h (g(x) - g(x+\tau))\,d\tau,$$

$$g_h'(x) = \frac{1}{h}[g(x+h) - g(x)].$$

which implies that g_h is continuously differentiable on $[0,+\infty)$ and g_h' is bounded on $[0,+\infty)$. Moreover

$$\sup_{[0,+\infty)} |g(x) - g_h(x)| \leqslant \omega(g,h), \qquad \sup_{[0,+\infty)} |g_h'(s)| \leqslant h^{-1}\omega(g,h). \tag{17}$$

Since

$$\begin{aligned} |V_\nu(g;x,t) - g(x)| &\leqslant |V_\nu\left(g - g_n; x,t\right)| + |V_\nu\left(g_n; x,t\right) - g_n(x)| + |g_n(x) - g(x)| \\ &= A + B + C, \end{aligned}$$

from (17) and the first part of this proof we get

$$B \leqslant h^{-1}\omega(g,h)\sqrt{8tx+16t^2(\nu+1)(\nu+2)}.$$

By (14) and (17) we have

$$\begin{aligned} A &= |V_\nu(g-g_h;x,t)| \leqslant V_\nu(|g-g_h|;x,t) \\ &\leqslant \sup_{[0,+\infty)} |g(s)-g_h(s)| \leqslant \omega(g,h), \\ C &\leqslant \omega(g,h). \end{aligned}$$

Hence

$$|V_\nu(g,x,t)-g(x)| \leqslant 2\omega(g,h) + h^{-1}\omega(g,h)\sqrt{8tx+16t^2(\nu+1)(\nu+2)}$$

for $x>0$, $t>0$.

Setting $h=\sqrt{8tx+16t^2(\nu+1)(\nu+2)}$ we obtain (16).

From Lemma 4 we have

THEOREM 5

If g is continuous on $[0,+\infty)$ and $\omega(g,h)<\infty$ for $h>0$, then

$$|V_\nu(f,r,t)-f(r)| \leqslant M\omega\left(g,\sqrt{8tr+16t^2(\nu+1)(\nu+2)}\right)$$

for $r>0$, $t>0$, where $g(s)=f(\sqrt{s})$, $s\in[0,+\infty)$.

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