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Global version of a Sternberg linearization theorem

Dedicated to Professor Zenon Moszner
on his 70th birthday

Abstract. Let $X(x) = DX(0)x + Z(x)$ be a smooth vector field in $\mathbb{R}^n$, with $X(0) = 0$, such that 0 is a contracting singular point of the linear part. We aim to extend the local Sternberg theorem on the linearization of X near the origin to a global version. It is done by assuming additionally that in some coordinates the sup norm $\|DZ\|_{\mathbb{R}^n}$ is finite and sufficiently small.

1. Introduction

Let $\mathbb{R}^n$ denote Euclidean n-space with inner product $\langle x, y\rangle$. Let $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Consider a C^∞ vector field

$$X(x) = Ax + Z(x) \quad \text{in } \mathbb{R}^n, \text{ with } X(0) = 0.$$

Let $A = DX(0)$ be the Jacobi matrix of X at 0.

Suppose that

$$\|DZ\| := \sup_{x \in \mathbb{R}^n} \{\|DZ(x)\|\}$$

is finite. It follows that X is globally Lipschitz and complete. Let $\exp tX$ be the X-flow defined for all $t \in \mathbb{R}^n$.

If f is a diffeomorphism of $\mathbb{R}^n$, the adjoint linear operator f_* acts on vector fields by

$$f_*X = Df \circ X \circ f^{-1}.$$

The vector fields X and f_*X are considered equivalent (or conjugate). In terms of differential equations, if $y = f(x)$ is meant as a change of variables,

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it transforms the differential system $x' = X(x)$ into the system $y' = (f_*X)(y)$. Therefore linearization theorems for vector fields refer equally to ordinary differential equations.

In [2] Sternberg proved the following theorem

THEOREM

Let X be a C^∞ vector field in a neighborhood of 0 in $\mathbb{R}^n$ with $X(0) = 0$. Let $A = DX(0)$ and suppose that

(E.1) *each eigenvalue λ_i of the matrix A satisfies $\operatorname{Re}\lambda_i < 0$,*

(E.2) $\lambda_j \neq \sum m_i \lambda_i$

for any non-negative integers m_i such that $\sum m_i > 1$.

Then there is a C^∞ local diffeomorphism F at 0 such that F_*X coincides with the linear part Ax of X in a neighborhood of 0.

The vector field X can be written as $X_0 + Z$ with $X_0(x) = Ax$ and $Z(x) = o(x)$, $|x| \to 0$. We ask for a global diffeomorphism of $\mathbb{R}^n$ which conjugates X with X_0.

(Equivalently, it would mean that the dynamical system $x' = Ax + Z(x)$ can be globally linearized to $y' = Ay$ by a smooth change of coordinates).

The special case considered in this note requires significantly weaker assumptions than those for the general theorems presented in author's paper [5].

2. Integral method

Our method to solve the global conjugacy problem is to apply the following technical proposition.

PROPOSITION

Let X, Z be C^k $(1 \leqslant k \leqslant \infty)$ complete vector fields on $\mathbb{R}^n$. Suppose that the integral

$$f = \mathrm{id} + \int_0^\infty T\exp(-tX).Z \circ \exp t(X+Z)\,dt \tag{1}$$

converges to class C^k $(k \geqslant 1)$ and

$$g = \mathrm{id} + \int_0^\infty T\exp(-t(X+Z)).Z \circ \exp tX\,dt \tag{2}$$

converges to class C^1 on $\mathbb{R}^n$

Then f and g are C^k diffeomorphisms of M, $g = f^{-1}$, and

$$f_*(X+Z) = X.$$

Proof. In the proof we use the Möller wave operator (Nelson [1]) which is known in quantum mechanics. Put $\phi_t = \exp tX$ and $\psi_t = \exp t(X+Z)$. Suppose that the diffeomorphisms $f_t = \phi_{-t} \circ \psi_t$ have the limit

$$\lim_{t\to\infty} \phi_{-t} \circ \psi_t = f \quad \text{(the wave operator).} \tag{3}$$

Then

$$\phi_{-t} f \psi_t = \lim_{s\to\infty} \phi_{-t}\phi_{-s}\psi_s\psi_t = \lim_{s\to\infty} \phi_{-(t+s)}\psi_{t+s} = f.$$

If f is invertible then $f \circ \psi_t \circ f^{-1} = \phi_t$. By differentiating in t we obtain $f_*(X+Z) = X$.

Differentiating f_t and $g_t = \psi_{-t} \circ \phi_t$ in t, and using the general identities

$$\phi_t' = X \circ \phi_t = D\phi_t.X,$$

we obtain easily

$$f_t' = (T\phi_{-t}.Z) \circ \psi_t \tag{4}$$

and

$$g_t' = (T\psi_{-t}.Z) \circ \phi_t\,. \tag{5}$$

The existence of both the limits $f = \lim_{t\to\infty} f_t$ and $g = \lim_{t\to\infty} g_t$ ensures the invertibility of f. Thus, we should integrate (4) and (5) in the interval [0,t] and pass to the limit as $t \to \infty$. It is well known that if f is C^k, for $k \geqslant 1$ and has a C^1 inverse g, then the inverse is C^k. This completes the proof of the proposition.

3. Main result

We shall prove the following

GLOBAL LINEARIZATION THEOREM

Let X be a C^∞ vector field in $\mathbb{R}^n$ with $X(0) = 0$ such that $DX(0)$ satisfies the eigenvalue conditions (E.1) *and* (E.2). *Suppose that the nonlinear part Z of X satisfies*

$$\|DZ\|_{\mathbb{R}^n} < \inf\{|\operatorname{Re}\lambda|; \quad \lambda \in \operatorname{Spect} DX(0)\}. \tag{6}$$

Then there is a C^∞ global diffeomorphism h such that h_*X coincides with the linear part of X in $\mathbb{R}^n$.

Proof. Let $X_0(x) = Ax$ be the linear part of X. As by the hypothesis $Z(0) = 0$ and the tangent map DZ is globally bounded, it follows from the Mean Value Theorem that for all $y \in \mathbb{R}^n$

$$\langle y, Z(y)\rangle \leqslant \|DZ\| |y|^2. \tag{7}$$

We have $\exp_t X_0(x) = e^{tA}x$. By the assumption, the real parts of the eigenvalues of A are contained in an interval $(-b, -c)$ for some $b \geqslant c > 0$. Then there is a norm in $\mathbb{R}^n$ such that

$$\|e^{tA}\| \leqslant e^{-ct}, \quad \|e^{-tA}\| \leqslant e^{bt}, \quad t \geqslant 0.$$

Since $\sup_{|y|=1}\langle y, Ay\rangle = \sup\{\mathrm{Re}\,(\mathrm{Spect}\, A)\}$, we have

$$\langle y, Ay\rangle \leqslant -c|y|^2, \quad y \in \mathbb{R}^n. \tag{8}$$

Because of (6) we also have $\|DZ\| < c$. Put $a := c - \|DZ\| > 0$. With $\phi(t) = \exp tX$ we have

$$\|\phi_t(x)\| \leqslant e^{-at}|x| \tag{9}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

Indeed, using (7) and (8) one obtains

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}|\phi_t(x)|^2 &= \langle \phi_t(x), (A+Z)\phi_t(x)\rangle \\ &\leqslant (-c + \|DZ\|)|\phi_t(x)|^2 \\ &= -a|\phi_t(x)|^2. \end{aligned}$$

Since $|\phi_0(x)| = |x|$, we get (9).

By the cited Sternberg theorem there exists a C^∞ diffeomorphism F, with $F(0) = 0$, such that $F_*X = X_0$ in a neighborhood U of the origin. Without loss of generality we may assume that $F - \mathrm{id}$ is globally bounded in $\mathbb{R}^n$ together with its tangent map $DF - I$ (for instance that it is of compact support). Hence also $F^{-1} - \mathrm{id}$ is globally bounded and there are $\alpha, \beta > 0$ such that $|F^{-1}(x)| \leqslant |x| + \alpha$, $\|DF\| \leqslant \beta$.

Define $\tilde{X} = F_*X = Ax + \tilde{Z}$. Then $\tilde{Z} = 0$ in a neighborhood of the origin and

$$\tilde{\psi}_t := \exp t\tilde{X} = F \circ \exp tX \circ F^{-1}.$$

Hence $\tilde{\psi}_t(0) = 0$ and by (9)

$$\|\tilde{\psi}_t(x)\| \leqslant \|DF\||(\exp tX)(F^{-1}(x))| \leqslant \|DF\|e^{-at}|F^{-1}(x)| \leqslant e^{-at}\beta(|x| + \alpha).$$

Now we are going to verify the convergence of the integrals of the Proposition in the version where X_0 stands for X and $\tilde{Z}$ stands for Z. Then the integrals (1) and (2) can be written in the form

$$f(x) = x + \int_0^\infty e^{-tA}.\tilde{Z}(\phi_t(x))\,dt$$

and

$$g(x) = x + \int_0^\infty D\phi_{-t}(e^{tA}x).\tilde{Z}(e^{tA}x)\,dt.$$

In both the integrals $\bar{Z}$ has inner vector functions which depend smoothly on (t, x) and tend to the origin as $t \to \infty$. Thus for any compact set K in $\mathbb{R}^n$ there exists a minimal time t_K such that for $t > t_K$ and $x \in K$ the arguments will take values in the neighborhood U where $\bar{Z}$ vanishes. Therefore both the integrals are finite and convergent uniformly with respect to x in any compact subset of $\mathbb{R}^n$. For the same reason, this is also true for all the derivatives $D^k f$ and $D^k g$. Hence f, g are C^∞ and $\lim f_t^{-1} = \lim g_t = g$ exists, so f^{-1} exists and is equal to g.

This completes the proof, since the composition $h = f \circ F$ is a diffeomorphism which conjugates X with X_0 in $\mathbb{R}^n$.

Remark

We see that if the nonlinear part Z of X has a sufficiently small global Lipschitz constant and X_0 satisfies (E.1), then the X-flow is globally asymptotically stable at the origin. This ensures that the local linearization (due to (E.2)) of X can be extended to a global one.

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