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On conjugacy of multivalent functions on the circle

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. We prove that every two expanding selfmappings of the circle which take each value in exactly n points are conjugate. The problem of the uniqueness of continuous conjugating functions is considered. Moreover, some applications to the determination of iterative roots of the above mappings are given.

A function f is said to be *n-valent* or shortly *n-to-1* if the pre-image of each its value consists of exactly n points. Let $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. In this note we shall consider n-to-1 mappings of the unit circle S^1 onto itself. The natural examples of n-to-1 mappings are the functions conjugate with monomials z^n. In this paper we consider the problem when n-to-1 functions conjugate with z^n. The problem of semi-conjugacy has been treated by M. Shub in [3] and reported in monograph [2].

Let us quote the following

Lemma 1 (see [4], [1])

Let $F : S^1 \longrightarrow S^1$ be a continuous function. Then there exist a unique integer n and a unique continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$F\left(e^{2\pi i t}\right) = e^{2\pi i f(t)}, \quad t \in \mathbb{R}, \tag{1}$$

$$f(t+1) = f(t) + n, \quad t \in \mathbb{R} \tag{2}$$

and

$$0 \leqslant f(0) < 1.$$

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The function f is said to be the *lift* of F and the integer n is called the *degree* of F ($\deg F = n$).

Let us begin with a few remarks on multivalent functions.

REMARK 1

Let $F : S^1 \longrightarrow S^1$ be continuous and the lift of F be strictly increasing (decreasing). Then F is n-to-1 ($n \neq 0$) if and only if $\deg F = n$ ($\deg F = -n$).

Proof. Suppose that $\deg F = n > 0$, $f(0) \leqslant \alpha < f(0) + 1$ and $w = e^{2\pi i\alpha}$. Since $f|_{[0,1)}$ is a bijection of $[0,1)$ onto $[f(0), f(0)+n)$, we infer that for every $k \in 0, \dots, n-1$ there exists a unique $t_k \in [0,1)$ such that $f(t_k) = \alpha + k$. Hence $F(z_k) = w$ for $z_k := e^{2\pi i t_k}$, $k = 0, \dots, n-1$ and $F(z) \neq w$ for $z \in S^1 \setminus \{z_0, \dots, z_{n-1}\}$.

Conversely, if F is n-to-1 and $\deg F = k$ then by the previous part F is k-to-1 so $k = n$.

The same is in the case of a decreasing lift.

REMARK 2

If $F : S^1 \longrightarrow S^1$ is continuous, $\deg F = n \neq 0$ *and F is $|n|$-to-1, then its lift is strictly monotonic.*

Proof. Let $n > 0$. For the indirect proof suppose that $f(t_1) = f(t_2) =: \alpha$ for some $t_1, t_2 \in [0,1)$ and $t_1 \neq t_2$. Put $a := e^{2\pi i\alpha}$ and $w := e^{2\pi i t_2}$. Write

$$[\alpha - f(0)] = ln + r, \quad \text{where } l, r \in \mathbb{Z} \text{ and } 0 \leqslant r < n.$$

The continuity of f and (2) imply that

$$f(0) + [ln, (l+1)n) \subset f[\,[l, l+1)\,].$$

Hence for every integer $k \in [-r, n-1-r]$ there exists $u_k \in [l, l+1)$ such that

$$f(u_k) = \alpha + k,$$

since $f(0) + ln \leqslant \alpha + k < f(0) + (l+1)n$. If $l = 0$, then we put $u_0 := t_1$. Define $a_k := e^{2\pi i u_k}$, where $k \in \{-r, \dots, n-1-r\}$. Obviously $a_{k_1} \neq a_{k_2}$, since $u_{k_1}, u_{k_2} \in [l, l+1)$ and $u_{k_1} \neq u_{k_2}$. Moreover, $w \neq a_k$ for all k. In fact, suppose $w = a_k$, for a $k \in \{-r, \dots, n-1-r\}$, then $e^{2\pi i t_2} = e^{2\pi i u_k}$. Since $t_2 \in [0,1)$ and $u_k \in [l, l+1)$, we have $t_2 = u_k + l$. Hence $\alpha = f(t_2) = f(u_k + l) = f(u_k) + nl = \alpha + k + nl$, so $k = -nl$. In view of the inequalities $-r \leqslant k \leqslant n-1-r$ and $0 \leqslant r < n$ we obtain that $l = k = 0$. Thus $t_2 = u_0$, but this is a contradiction, since $u_0 := t_1$, if $l = 0$.

Let us note that

$$F(a_k) = e^{2\pi i f(u_k)} = e^{2\pi i\alpha} = a.$$

and

$$F(w) = e^{2\pi i f(t_2)} = e^{2\pi i\alpha} = a,$$

so $w, a_k \in F^{-1}[\{a\}]$ for $k \in \{-r, \dots, n-1-r\}$. Thus the pre-image $F^{-1}[\{a\}]$ contains at least $n+1$ points. This is a contradiction, so f is strictly increasing in $[0,1)$ and in view of (2) f is strictly incresing in $\mathbb{R}$.

The same proof is in the case of negative degree.

REMARK 3

There exist continuous n-to-1 functions $F : S^1 \longrightarrow S^1$ $(n > 1)$ of degree one.

We give an example for $n = 3$. Let $f : [0,1] \longrightarrow [0,1]$ be the piecewise linear function with the vertices: $(0,0)$, $(\frac{1}{5}, \frac{1}{2})$, $(\frac{2}{5}, 0)$, $(\frac{3}{5}, 1)$, $(\frac{4}{5}, \frac{1}{2})$, $(1,1)$. It is easy to verify that $F\left(e^{2\pi i t}\right) := e^{2\pi i f(t)}$ is 3-to-1 function and $\deg F = 1$.

Define two classes of functions

$$\begin{aligned}\mathcal{K}_n := \{F : S^1 \longrightarrow S^1 :\ & F \textit{ is continuous}, \deg F = n,\\ & |x-y| < |f(x) - f(y)|,\ x \neq y,\ x, y \in \mathbb{R}\},\end{aligned}$$

where f is the lift of F and

$$\begin{aligned}\mathcal{K}_n^* := \{F : S^1 \longrightarrow S^1 :\ & F \textit{ is continuous}, \deg F = n,\\ & |x-y| \leqslant \gamma(|f(x)-f(y)|),\ x, y \in \mathbb{R},\ \textit{for an increasing}\\ & \textit{function } \gamma : [0,\infty) \longrightarrow [0,\infty) \textit{ such that its sequence}\\ & \textit{of iterates } \gamma^m \textit{ converges pointwisely to } 0\}.\end{aligned}$$

It is easy to see that $\mathcal{K}_n^* \subset \mathcal{K}_n$.

REMARK 4

If $F \in \mathcal{K}_n$, then F has exactly $|n| - 1$ fixed points.

Proof. Put $h(t) = f(t) - t$. Let $z_0 = e^{2\pi i t_0}$ and $t_0 \in [0,1)$. Let us note that $F(z_0) = z_0$ if and only if $h(t_0) \in \mathbb{Z}$. Assume that F preserves orientation. It is easy to see that h is an increasing homeomorphism and

$$h[\,[0,1)\,] = [h(0), h(1)) = [f(0), f(0) + n - 1).$$

We have

$$\mathbb{Z} \cap [f(0), f(0) + n - 1) = \begin{cases} \{0, \dots, n-2\}, & \text{if } f(0) = 0,\\ \{1, \dots, n-1\}, & \text{if } f(0) > 0.\end{cases}$$

Thus, if $f(0) = 0$, $\mathbb{Z} \cap h[\,[0,1)\,] = \{0, \ldots, n-2\}$ and if $f(0) > 0$, $\mathbb{Z} \cap h[\,[0,1)\,] = \{1, \ldots, n-1\}$, whence it follows that F has exactly $n-1$ fixed points.

If F reverses orientation, the proof is analogous.

REMARK 5

Let $F : S^1 \longrightarrow S^1$ be continuous, $0 \leqslant \alpha < 1$, $F_a(z) := a^{-1}F(az)$, $z \in S^1$, where $a = e^{2\pi i\alpha}$ and let f and f_a be the lifts of F and F_a. Then

(i) $f_a(t) = f(t+\alpha) - \alpha - [f(\alpha) - \alpha]$, $t \in \mathbb{R}$.

(ii) $F(a) = a \iff F_a(1) = 1 \iff f_a(0) = 0$.

(iii) $F \in \mathcal{K}_n \iff F_a \in \mathcal{K}_n$.

Proof. (i) We have

$$F_a\left(e^{2\pi it}\right) = e^{-2\pi i\alpha} F\left(e^{2\pi i(\alpha+t)}\right) = e^{2\pi i(f(\alpha+t)-\alpha)}.$$

In view of the continuity of f_a and f it follows that $f_a(t) = f(t+\alpha) - \alpha - k$ for a $k \in \mathbb{Z}$. Since $f_a(0) \in [0,1)$ we have $f(\alpha) - \alpha - k \in [0,1)$. Thus $[f(\alpha) - \alpha] = k$.

(ii) The first equivalence is trivial. Let us note that the equality $0 = f_a(0) = f(\alpha) - \alpha - [f(\alpha) - \alpha]$ is equivalent to the condition $f(\alpha) - \alpha \in \mathbb{Z}$. Since $F(a) = e^{2\pi i f(\alpha)}$ and $a = e^{2\pi i\alpha}$ we get (ii).

(iii) It is a simple consequence of (i) and (2).

Directly from the definition of the degree we obtain the following

REMARK 6

If $F, G : S^1 \longrightarrow S^1$ are continuous, then $\deg F \circ G = \deg F \deg G$.

To solve the problem posed in the introduction we consider a more general problem of conjugacy of functions from the class $\mathcal{K}_n$.

The main tool in the further part of the paper are the following propositions:

PROPOSITION 1 (see [5])

Let J be a closed and finite interval with the ends a and b. Assume that:

(H$_1$) *$f_0, \ldots, f_{n-1} : [0,1] \longrightarrow [0,1]$ are continuous, strictly increasing mappings and $f_0(0) = 0$, $f_{n-1}(1) = 1$, $f_{k+1}(0) = f_k(1)$, $k = 0, \ldots, n-2$;*

(H$_2$) *$g_0, \ldots, g_{n-1} : J \longrightarrow J$ are continuous, strictly increasing mappings and $g_0(a) = a$, $g_{n-1}(b) = b$, $g_{k+1}(a) = g_k(b)$, $k = 0, \ldots, n-2$.*

If

$$|f_k(x) - f_k(y)| < |x - y|, \quad x \neq y, \ x, y \in [0,1], \ k = 0, \ldots, n-1 \tag{3}$$

and

$$|g_k(x) - g_k(y)| < |x - y|, \quad x \neq y,\ x, y \in J,\ k = 0, \dots, n-1, \tag{4}$$

then the system

$$\varphi(f_k(x)) = g_k(\varphi(x)), \quad x \in [0,1],\ k = 0, \dots, n-1 \tag{5}$$

has a unique solution $\varphi : [0,1] \longrightarrow J$. *This solution* φ *is continuous and strictly monotonic with* $\varphi(0) = a$ *and* $\varphi(1) = b$.

PROPOSITION 2 (see [5])

Let $f_0, \dots, f_{n-1}$ *satisfy* (H_1) *and* $g_0, \dots, g_{n-1}$ *satisfy* (H_2) *with* $J = \mathbb{R}$, *for some* $a, b \in \mathbb{R}$. *Suppose that*

$$|g_k(x) - g_k(y)| \leqslant \gamma(|x - y|), \quad x, y \in \mathbb{R},\ k = 0, \dots, n-1, \tag{6}$$

where $\gamma : [0, \infty) \longrightarrow [0, \infty)$ *is an increasing function such that its sequence of iterates* γ^n *converges pointwisely to* 0. *Then system* (5) *has a unique bounded solution. This solution is continuous and monotonic.*

We shall prove simultaneously the following two theorems:

THEOREM 1

If $F, G \in \mathcal{K}_n$ $(n \geqslant 2)$, $F(a) = a$, $G(b) = b$, *then for every integer* $r \neq 0$ *there exists a unique continuous solution* $\Phi : S^1 \longrightarrow S^1$ *of the equation*

$$\Phi(F(z)) = G(\Phi(z)), \quad z \in S^1, \tag{7}$$

such that $\deg \Phi = r$, $\Phi(a) = b$ *and* $\varphi[\,[0,1]\,] = [\varphi(0), \varphi(1)]$, *where* φ *is the lift of* Φ. *This lift is strictly monotonic. Moreover,* Φ *is a homeomorphism if and only if* $|r| = 1$.

THEOREM 2

If $F \in \mathcal{K}_n$, $G \in \mathcal{K}_n^*$ $(n \geqslant 2)$, $F(a) = a$, $G(b) = b$, *then for every integer* $r \neq 0$, *there exists a unique continuous solution* $\Phi : S^1 \longrightarrow S^1$ *of equation* (7) *such that* $\deg \Phi = r$ *and* $\Phi(a) = b$. *The lift of* Φ *is strictly monotonic.*

Proof. Let $F, G \in \mathcal{K}_n$, $F(a) = a$, $G(b) = b$. Put $F_a(z) := a^{-1}F(az)$ and $G_b(z) = b^{-1}G(bz)$, $z \in S^1$. Let us note that if a function $\Phi : S^1 \longrightarrow S^1$ satisfies (7) and $\Phi(a) = b$, then

$$\Phi_{a,b}(z) := b^{-1}\Phi(az)$$

satisfies

$$\Phi_{a,b}(F_a(z)) = G_b(\Phi_{a,b}(z)), \quad z \in S^1 \tag{8}$$

and $\Phi_{a,b}(1) = 1$. Conversely, if $\Psi : S^1 \longrightarrow S^1$ satisfies (8) and $\Psi(1) = 1$, then $\Phi(z) = b\Psi(a^{-1}z)$ satisfies (7) and $\Phi(a) = b$.

Thus in view of Remark 5 (iii) we may assume further that $a = b = 1$.

First we shall prove the uniqueness. Let $\Phi : S^1 \longrightarrow S^1$ be continuous solution of (7) such that $\deg \Phi = r$, $\Phi(1) = 1$ and φ be the lift of Φ. We have $f(0) = 0$, $g(0) = 0$ and $\varphi(0) = 0$. Moreover,

$$g(t+1) = g(t) + n, \quad t \in \mathbb{R} \tag{9}$$

and

$$\varphi(t+1) = \varphi(t) + r, \quad t \in \mathbb{R}. \tag{10}$$

We have

$$\Phi\left(F\left(e^{2\pi i t}\right)\right) = \Phi\left(e^{2\pi i f(t)}\right) = e^{2\pi i \varphi(f(t))}, \quad t \in \mathbb{R}$$

and

$$G\left(\Phi\left(e^{2\pi i t}\right)\right) = G\left(e^{2\pi i \varphi(t)}\right) = e^{2\pi i g(\varphi(t))}, \quad t \in \mathbb{R}.$$

Hence in view of (7)

$$\varphi(f(t)) - g(\varphi(t)) \in \mathbb{Z}, \quad \text{for } t \in \mathbb{R}. \tag{11}$$

Since φ, f and g are continuous we infer that there exists a $k \in \mathbb{Z}$ such that

$$\varphi(f(t)) - g(\varphi(t)) = k, \quad \text{for } t \in \mathbb{R}.$$

We have $k = 0$, because $\varphi(f(0)) = g(\varphi(0))$. Thus

$$\varphi(f(t)) = g(\varphi(t)), \quad t \in \mathbb{R}. \tag{12}$$

Put $J_r := [0, r]$, if $r > 0$ and $J_r := [r, 0]$, if $r < 0$.

Define

$$f_i(t) := f^{-1}(t+i), \quad t \in [0,1],\ i = 0, \dots, n-1$$

and

$$g_i(t) := g^{-1}(t+ir), \quad t \in \mathbb{R},\ i = 0, \dots, n-1.$$

We have

$$f(f_i(t)) = t + i, \quad \text{for } t \in [0,1],$$

so by (12) and (10) we get

$$\varphi(t) + ir = g(\varphi(f_i(t))), \quad t \in [0,1],\ i = 0, \dots, n-1$$

and

$$g^{-1}(\varphi(t) + ir) = \varphi(f_i(t)), \quad t \in [0,1],\ i = 0, \dots, n-1.$$

The last equalities we may write in the form of equation (5).

It is easy to verify that $f_0, \dots, f_{n-1}$ satisfy (H$_1$) and $g_0, \dots, g_{n-1}$ satisfy (H$_2$) with $a = 0$ and $b = r$. Moreover, the lifts f and g are invertible and

$$|f^{-1}(x) - f^{-1}(y)| < |x - y| \text{ and } |g^{-1}(x) - g^{-1}(y))| < |x - y|, \quad x \neq y,\ x, y \in \mathbb{R},$$

because $F, G \in \mathcal{K}_n$, whence it follows that $f_0, \dots, f_{n-1}$ satisfy (3) and $g_0, \dots, g_{n-1}$ satisfy (4). Furthermore, if we assume that $G \in \mathcal{K}_n^*$, thent

$$|g^{-1}(x) - g^{-1}(y)| \leqslant \gamma(|x-y|), \quad x, y \in \mathbb{R},$$

for an increasing function $\gamma : [0, \infty) \longrightarrow [0, \infty)$ such that $\gamma^n(t) \longrightarrow 0$, for $t \geqslant 0$. Hence it follows that $g_0, \dots, g_{n-1}$ satisfy (6).

In view of Proposition 1 system (5) has a unique solution $\overline{\varphi} : [0,1] \longrightarrow J_r$. If Φ is a continuous solution of (7) which lift φ satisfies the condition $\varphi[\,[0,1]\,] = [\varphi(0), \varphi(1)] = J_r$, then by Proposition 1 $\overline{\varphi} = \varphi|_{[0,1]}$. However, if $F \in \mathcal{K}_n$, $G \in \mathcal{K}_n^*$ and Φ is a continuonus solution of (7) of degree r such that $\Phi(1) = 1$, then by Proposition 2 its lift φ is monotonic, so $\varphi[\,[0,1]\,] = J_r$ and consequently $\overline{\varphi} = \varphi|_{[0,1]}$. Hence we infer that if $F \in \mathcal{K}_n$ and $G \in \mathcal{K}_n^*$ ($F, G \in \mathcal{K}_n$) then equation (7) has at most one continuous solution Φ of degree r such that $\Phi(1) = 1$ (and $\varphi[\,[0,1]\,] = [\varphi(0), \varphi(1)]$).

Conversely, by Proposition 1 solution $\overline{\varphi}$ of (7) is continuous, strictly monotonic, $\overline{\varphi}(0) = 0$ and $\overline{\varphi}(1) = r$. Let us note that the function $\psi(t) := \overline{\varphi}(t - [t]) + r[t]$, for $t \in \mathbb{R}$ is continuous, strictly monotonic, $\psi|_{[0,1]} = \overline{\varphi}$ and fulfils (10). Hence

$$\overline{\varphi}(t) + ir = \psi(t) + ir = \psi(t+i) = \psi(f(f_i(t))), \quad t \in [0,1].$$

On the other hand by (5)

$$\overline{\varphi}(t) = g_i^{-1}(\overline{\varphi}(f_i(t))), \quad t \in [0,1], \ i = 0, \dots, n-1.$$

Then

$$\psi(f(f_i(t))) = g(\overline{\varphi}(f_i(t))) = g(\psi(f_i(t))), \quad t \in [0,1], \ i = 0, \dots, n-1. \tag{13}$$

It follows from (H_1) that

$$\bigcup_{i=0}^{n-1} f_i[\,[0,1]\,] = \bigcup_{i=0}^{n-1} [f_i(0), f_i(1)] = [f_0(0), f_{n-1}(1)] = [0,1].$$

Hence in view of (13) we obtain that

$$\psi(f(t)) = g(\psi(t)), \quad t \in [0,1],$$

so the function $\Psi\left(e^{2\pi i t}\right) := e^{2\pi i \psi(t)}$, $t \in [0,1]$ is a continuous solution of (7), such that $\psi(1) = 1$ and $\deg \Psi = r$.

The remaining part of thesis is a simple consequence of Remark 1, since the lift of Φ is strictly monotonic.

Let $a, b \in S^1$, $a \neq b$. Then there exist $t_a, t_b \in \mathbb{R}$ such that $t_a < t_b < t_a + 1$, $a = e^{2\pi i t_a}$ and $b = e^{2\pi i t_b}$. Put

$$\operatorname{arc}(a,b) := \{e^{2\pi i t} : \ t \in (t_a, t_b)\}.$$

THEOREM 3

Let $F, G \in \mathcal{K}_n$ $(n \geqslant 2)$, $a \in S^1$, $F(a_k) = a$, for $k = 0, \ldots, n$, $a_0 = a_n = a$ and

$$\operatorname{Arg} \frac{a_{k+1}}{a_0} > \operatorname{Arg} \frac{a_k}{a_0}, \quad k = 0, \ldots, n-2.$$

A solution $\Phi : S^1 \longrightarrow S^1$ of equation (7) is continuous and of degree one if and only if

$$\operatorname{Arg} \frac{\Phi(a_{k+1})}{\Phi(a_0)} > \operatorname{Arg} \frac{\Phi(a_k)}{\Phi(a_0)}, \quad k = 0, \ldots, n-2 \tag{14}$$

and

$$\Phi[\operatorname{arc}(a_k, a_{k+1})] \subset \operatorname{arc}(\Phi(a_k), \Phi(a_{k+1})). \tag{15}$$

Proof. Let Φ satisfy (7), (14) and (15). Then $\Phi_{a,b}$, where $b = \Phi(a)$ satisfies (8) and

$$\Phi_{a,b}\left[\operatorname{arc}\left(\frac{a_k}{a_0}, \frac{a_{k+1}}{a_0}\right)\right] \subset \operatorname{arc}\left(\Phi_{a,b}\left(\frac{a_k}{a_0}\right), \Phi_{a,b}\left(\frac{a_{k+1}}{a_0}\right)\right), \quad k = 0, \ldots, n-1.$$

Thus we may assume that $a = 1$, $F(1) = 1$, $G(1) = 1$ and $\Phi(1) = 1$. Put $b_k := \Phi(a_k)$, $k = 0, \ldots, n$. We have

$$G(b_k) = G(\Phi(a_k)) = \Phi(F(a_k)) = \Phi(1) = 1.$$

It follows from (14) that

$$\operatorname{Arg} b_{k+1} > \operatorname{Arg} b_k, \quad \text{for } k = 0, \ldots, n-2.$$

Put

$$s_k := \frac{\operatorname{Arg} a_k}{2\pi}, \quad t_k := \frac{\operatorname{Arg} b_k}{2\pi}, \quad \text{for } k = 0, \ldots, n-1 \text{ and } s_n := t_n := 1.$$

Let us note that $f(s_k), g(t_k) \in \mathbb{Z}$, for $k = 0, \ldots, n$, where f and g are the lifts of F and G. Since f is strictly increasing, $f(0) = 0$ and $f(1) = n$, we have

$$0 = f(s_0) < f(s_1) < \ldots < f(s_{n-1}) < f(s_n) = n,$$

so $f(s_k) = k$, for $k = 0, \ldots, n$. Similarly $g(t_k) = k$, for $k = 0, \ldots, n$.

Let $\overline{\varphi} : [0, 1) \longrightarrow [0, 1)$ be a mapping such that

$$\Phi\left(e^{2\pi it}\right) = e^{2\pi i \overline{\varphi}(t)}, \quad t \in [0, 1).$$

Define

$$\varphi(t) := \overline{\varphi}(t - [t]) + [t], \quad t \in \mathbb{R}.$$

Obviously

$$\Phi\left(e^{2\pi it}\right) = e^{2\pi i\varphi(t)}, \quad t \in \mathbb{R},$$

$$\varphi(t+1) = \varphi(t) + 1, \quad t \in \mathbb{R} \tag{16}$$

and φ fulfils (11).

Let $t \in [0,1)$, then $t \in [s_k, s_{k+1}) =: I_k$, for a $k \in \{0, \dots, n-1\}$. In view of (15)

$$e^{2\pi i\varphi(t)} = \Phi\left(e^{2\pi it}\right) \in \operatorname{arc}[b_k, b_{k+1}) = \operatorname{arc}[e^{2\pi it_k}, e^{2\pi it_{k+1}}),$$

so $\varphi(t) \in [t_k, t_{k+1}) =: J_k$. Consequently, $g(\varphi(t)) \in g[J_k] = [k, k+1)$. On the other hand, $f(t) \in f[I_k] = [k, k+1)$ and $\varphi(f(t)) \in \varphi[\,[k, k+1)\,] = [k, k+1)$. Hence $|g(\varphi(t)) - \varphi(f(t))| < 1$. Taking into account (11) we get

$$\varphi(f(t)) = g(\varphi(t)), \quad \text{for} \ \ t \in [0,1),$$

so by (2), (9) and (16) φ satisfies (12).

Further, similarly as in the previous proof one can verify that $\varphi|_{[0,1]}$ satisfies system (5). By Proposition 1 φ is continuous, strictly increasing and $\varphi[[0,1]] = [0,1]$. Thus Φ is continuous and $\deg \Phi = 1$.

Conversely, if a solution Φ is continuous and $\deg \Phi = 1$, then its lift φ is continuous, strictly increasing and $\varphi[[0,1)] = [0,1)$. Hence

$$\begin{aligned}\Phi[\operatorname{arc}(a_k, a_{k+1})\,] &= \{e^{2\pi i\varphi(t)} : \ t \in (s_k, s_{k+1})\} = \{e^{2\pi iu} : \ u \in \varphi[\,(s_k, s_{k+1})\,]\,\} \\ &= \{e^{2\pi iu} : \ u \in (\varphi(s_k), \varphi(s_{k+1}))\} \\ &= \operatorname{arc}(\Phi(a_k), \Phi(a_{k+1})), \quad k = 0, \dots, n-1.\end{aligned}$$

Directly from Theorem 2 we obtain the following main result:

THEOREM 4

If $G \in \mathcal{K}_n$ $(n \geqslant 2)$, then for every $a \in S^1$ such that $G(a) = a$ and every $r \in \mathbb{Z} \setminus \{0\}$ there exists a unique continuous solution Φ of degree r of the equation

$$\Phi(z^n) = G(\Phi(z)), \quad z \in S^1, \tag{17}$$

such that $\Phi(1) = a$. If $|r| = 1$, then this solution is a homeomorphism.

REMARK 7

If $G \in \mathcal{K}_n$ $(n \geqslant 2)$ and Φ is a continuous solution of (17) *of degree r, then the remaining continuous solutions of* (17) *of degree r are given by the formula*

$$\Phi_k(z) = \Phi\left(q^k z\right), \quad z \in S^1, \ k = 1, \dots, n-2, \tag{18}$$

where $q = e^{\frac{2\pi i}{n-1}}$.

Proof. Let us note that every solution Φ of equation (17) has the property that $\Phi(1)$ is a fixed point of G. Thus G has exactly $n-1$ fixed points (cf. Remark 4). Hence by Theorem 4 equation (17) has exactly $n-1$ continuous solutions of degree r. Let us note that Φ_k for $k = 0, \dots, n-2$ satisfy (17). In fact, $q^n = q$, so

$$\Phi_k(z^n) = \Phi\left(q^k z^n\right) = \Phi\left(\left(q^k\right)^n z^n\right) = \Phi\left(\left(q^k z\right)^n\right)$$
$$= G\left(\Phi\left(q^k z\right)\right) = G(\Phi_k(z)), \quad z \in S^1.$$

By Theorem 4 and Remark 7 we get

COROLLARY 1

If $\Phi : S^1 \longrightarrow S^1$ is continuous and

$$\Phi(z^n) = \Phi(z)^n, \quad z \in S^1,$$

then $\Phi(z) = q^k z^r$, for a $k \in \{0, \dots, n-2\}$ and an $r \in \mathbb{Z} \setminus \{0\}$.

By Theorem 2 we have

COROLLARY 2

If $F \in \mathcal{K}_n^*$ $(n \geqslant 2)$, $F(a) = a$ and $\Phi : S^1 \longrightarrow S^1$ is a continuous solution of the equation

$$\Phi(F(z)) = F(\Phi(z)), \quad z \in S^1,$$

such that $\Phi(a) = a$ and $\deg \Phi = 1$, then $\Phi(z) = z$, $z \in S^1$.

Finally we shall show some applications of Theorem 4 to the determination the iterative roots.

A function H is said to be an *iterative root of order k* of a function G if $H^k = G$, where H^k denotes the k-th iterate of H. We have

THEOREM 5

A function $G \in \mathcal{K}_n$ $(n \geqslant 2)$ has a continuous iterative root of order k iff $n = r^k$ for an integer r.

Proof. Suppose $H : S^1 \longrightarrow S^1$ is continuous, $\deg H = r$ and

$$H^k(z) = G(z), \quad z \in S^1$$

By Remark 6 $\deg H^k = r^k$, so $r^k = \deg G = n$.

Conversely, in view of Theorem 4 there exists a homeomorphism $\Phi : S^1 \longrightarrow S^1$ fulfilling (17). Put $H(z) = \Phi(\Phi^{-1}(z)^r)$. We have

$$H^k(z) := \Phi\left(\Phi^{-1}(z)^{r^k}\right) = \Phi\left(\Phi^{-1}(z)^n\right) = G(z), \quad z \in S^1.$$

Denote by $\mathcal{S}_n$ the class of all functions conjugated with the monomial z^n, that is

$$\mathcal{S}_n := \{\Psi : S^1 \longrightarrow S^1 : \ \Psi(z) = \Phi\left(\Phi^{-1}(z)^n\right), \ z \in S^1,$$
$$\textit{where } \Phi : S^1 \longrightarrow S^1 \textit{ is a homeomorphism}\}.$$

Let us note that $\mathcal{K}_n \subset \mathcal{S}_n$.

We shall show

THEOREM 6

If $G \in \mathcal{K}_n$ $(n \geqslant 2)$ and $n = r^k$, then G has exactly $\frac{n-1}{r-1}$ iterative roots of order k in the class $\mathcal{S}_r$. They are given by the formula

$$H(z) = \Phi\left(p^j \Phi^{-1}(z)^r\right), \quad z \in S^1,\ j = 0, \ldots, \tfrac{n-r}{r-1}, \tag{19}$$

where $p = e^{\frac{2\pi i(r-1)}{n-1}}$ and Φ is a homeomorphic solution of equation (17).

Proof. Let Φ be a homeomorphic solution of (17) and H be given by (19). Then

$$H^k(z) = \Phi\left(p^j p^{jr} \ldots p^{jr^{k-1}} \Phi^{-1}(z)^{r^k}\right) = \Phi\left(p^{j\frac{n-1}{r-1}} \Phi^{-1}(z)^n\right) = \Phi\left(\Phi^{-1}(z)^n\right)$$
$$= G(z),$$

since $p^{\frac{n-1}{r-1}} = 1$.

By Remark 7 the function $\Psi(z) := \Phi(q^{n-1-j}z)$, where $q := e^{\frac{2\pi i}{n-1}}$ is a homeomorphic solution of (17) and

$$\begin{aligned}\Psi\left(\Psi^{-1}(z)^r\right) &= \Phi\left(q^{n-1-j}\left[q^{j+1-n}\Phi^{-1}(z)\right]^r\right)\\ &= \Phi\left(q^{(n-1-j)(1-r)}\Phi^{-1}(z)^r\right) = \Phi\left(p^j\Phi^{-1}(z)^r\right)\\ &= H(z),\end{aligned} \tag{20}$$

since $q^{r-1} = p$. Therefore $H \in \mathcal{S}_r$.

Suppose now that $H \in \mathcal{S}_r$ and $H^k = G$. Then there exists a homeomorphism $\Psi : S^1 \longrightarrow S^1$ such that

$$H(z) = \Psi\left(\Psi^{-1}(z)^r\right), \quad z \in S^1. \tag{21}$$

Hence

$$G(\Psi(z)) = H^k(\Psi(z)) = \Psi\left(z^{r^k}\right) = \Psi(z^n), \quad z \in S^1, \tag{22}$$

so, in view of Remark 7 $\Psi(z) = \Phi(q^j z)$ for a $j \in \{0, \ldots, n-1\}$. Further by (21) and (22) we get (19), because $j = l\frac{n-1}{r-1} + \bar{j}$, where $\bar{j} \in \{0, \ldots, \frac{n-1}{r-1} - 1\}$ and $p^j = p^{\bar{j}}$.

COROLLARY 3

Let $n = r^k$. The functions $H(z) = pz^r$, where $p^{\frac{n-1}{r-1}} = 1$ are the only solutions of the equation

$$H^k(z) = z^n, \quad z \in S^1$$

in the class $\mathcal{K}_r$.

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