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# Annales Universitatis Paedagogicae Cracoviensis

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## Studia Mathematica VIII

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## Studia Mathematica VIII

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# Explicit solution of a class of Riemann–Hilbert problems

**Abstract.** Analytical solutions to a special class of Riemann–Hilbert boundary value problems on multiply connected domains are presented. The solutions are expressed, up to a finite number of accessory parameters, as non-singular indefinite integrals whose integrands are expressed in terms of the Schottky–Klein prime function associated with the Schottky double of the planar domain.

## 1. A class of Riemann–Hilbert problems

The subject of this paper is a special class of Riemann–Hilbert problems (RH problems) on multiply connected planar domains. The study of general RH problems is a classical subject and discussions of it can be found in standard monographs on boundary value problems [9], [18], [13]. A solution of the general (Riemann)–Hilbert boundary value problem has been found, using successive iteration methods, by Mityushev [14]. Here we restrict attention to a special (but important) subclass of the same RH problems and find an analytical expression for the solutions, up to a finite set of accessory parameters, in terms of a transcendental function known as the Schottky–Klein prime function [3] associated with the multiply connected domain.

We define a *circular domain* $D_\zeta$ in a complex parametric $\zeta$-plane to be a domain whose boundaries are all circles. Let $D_\zeta$ be the $M+1$ connected circular domain in a $\zeta$-plane consisting of the unit disc with $M$ smaller discs excised from its interior. The outer boundary of $D_\zeta$ is the unit circle which we label $C_0$. Label the $M$ inner boundary circles of $D_\zeta$ as $C_1, \dots, C_M$. For $k = 0, 1, \dots, M$ let the centre and radius of $C_k$ be $\delta_k$ and $q_k$ respectively.

Consider the Riemann–Hilbert problem for the function $w(\zeta)$:

$$\mathrm{Re}\left[\overline{\lambda_k}\, w(\zeta)\right] = d_k \qquad \text{on } C_k,\ k = 0, 1, \dots, M, \tag{1}$$

where $\{\lambda_k \in \mathbb{C} \mid |\lambda_k| = 1,\ k = 0, 1, \dots, M\}$ is a set of complex constants with unit modulus and $\{d_k \in \mathbb{R} \mid k = 0, 1, \dots, M\}$ is a set of real constants. We solve for

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$w(\zeta)$ satisfying (1) that is analytic, but not necessarily single-valued, in $D_\zeta$ except for a simple pole, with known residue, at some point $\zeta = \beta$ strictly inside $D_\zeta$.

Circular domains are a canonical class of planar domains because every planar domain is conformally equivalent to some circular domain [10]. Because of this, and because the class of RH problems (1) is conformally invariant, it means that the solution scheme which follows is rather general. It applies, up to conformal mapping from the canonical class of circular domains, to *any* multiply connected planar region.

Problem (1) is a generalization of the classical Schwarz problem [9], [18], [13], a case of which is retrieved on making the choice, for example, that $\lambda_k = 1$ for all $k = 0, 1, \ldots, M$ in (1). This paper produces an analytical expression for the solution of (1) when the constants $\{\lambda_k \in \mathbb{C} \mid k = 0, 1, \ldots, M\}$ are generally distinct. The solution is expressed as a non-singular, indefinite integral whose integrand is written in terms of the Schottky–Klein prime function [3] associated with $D_\zeta$. This integrand depends on a finite set of accessory parameters that can, in principle, be determined (for example, numerically) from the given data $\{\lambda_k, d_k \in \mathbb{C} \mid k = 0, 1, \ldots, M\}$.

The special form of RH problem (1) has been considered by other authors. Vekua [18] shows that, if it exists, the solution of the RH problem (1) is unique [18]. Wegmann & Nasser [19] study the doubly connected case $M = 1$ of (1) in a recent paper on numerical solutions of RH problems on multiply connected regions using integral equations based on the generalized Neumann kernel.

The class of RH problems appears in a variety of applications, especially in the more general (discontinuous) case when the value of the constant $\lambda_k$ assumes different values on different segments of the circle $C_k$ (the methods of this paper, presented for the continuous problem, can be generalized to this case). One of the more important applications is to free streamline theory in hydrodynamics. There, in the study of jets and cavities, it is traditional to study a function known as the Joukowski function [11], often written as

$$\Omega(\zeta) \equiv \log\left(\frac{1}{V_0}\frac{dw(z)}{dz}\right),$$

where $z = x + \mathrm{i}y$, $V_0$ is a constant scaling factor and $w(z)$ is an analytic function in the flow region (known as the complex velocity potential). On any solid boundaries in contact with the fluid, the imaginary part of $\Omega(\zeta)$ is constant; on any free streamlines, owing to the constancy of pressure in a cavity region on one side of the free streamline and Bernoulli's theorem, it is the real part of $\Omega(\zeta)$ that is constant. Since a single streamline in a real flow can, in part, be in contact with a solid boundary and then separate into a free streamline bounding a cavity, $\Omega(\zeta)$ turns out to satisfy a (discontinuous) Riemann–Hilbert problem of precisely the form (1). In the simply connected situation, Schwarz–Christoffel methods have proved to be very useful in problems of this kind [11]. Interestingly, there has been recent interest [2] in developing this nonlinear theory to flows involving multiple body-cavity systems. The theory presented here, for multiply connected situations, should find application in such studies.

## 2. Function theory

The investigation we now present borrows ideas from prior work by the author [5], [6] in which new analytical formulae for the Schwarz–Christoffel mappings to bounded and unbounded polygonal domains were constructed. Although this viewpoint is not the one taken in [5], [6], such Schwarz–Christoffel mappings can be viewed as satisfying a RH problem on a multiply connected domain of exactly the form (1). Here, the same constructive method is exploited to find explicit representations of the solution of broader classes of RH problems in multiply connected domains.

In this paper, for ease of exposition, we focus on the continuous case where the constant $\lambda_k$ assumes the same value at all points on the circle $C_k$ (in the discontinuous analogue, which is more akin to the usual Schwarz–Christoffel problem, the value of this constant is allowed to be different on different segments of $C_k$). A consequence of this assumption is that we effectively do not allow any branch point singularities of $w(\zeta)$ on any of the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. The method, however, can be readily generalized to the case where branch points *are* present.

We now construct some special functions associated with $D_\zeta$. First, for $k = 0, 1, \ldots, M$, define the Möbius transformation $\phi_k(\zeta)$ by

$$\phi_k(\zeta) = \overline{\delta_k} + \frac{q_k^2}{\zeta - \delta_k}, \qquad k = 0, 1, \ldots, M. \tag{2}$$

It is straightforward to check that for $\zeta$ on circle $C_k$,

$$\phi_k(\zeta) = \overline{\zeta}.$$

We define the *reflection* of a point $\zeta$ in the circle $C_k$ by $\overline{\phi_k(\zeta)}$. Then, for $k = 1, \ldots, M$, introduce the Möbius transformation $\theta_k(\zeta)$ defined by

$$\theta_k(\zeta) = \overline{\phi_k\left(\overline{\zeta}^{-1}\right)}, \qquad k = 1, \ldots, M. \tag{3}$$

It follows from (3) and (2) that

$$\theta_k(\zeta) = \delta_k + \frac{q_k^2 \zeta}{1 - \overline{\delta_k}\zeta}, \qquad k = 1, \ldots, M.$$

For $k = 1, \ldots, M$, let $C_k'$ denote the reflection of $C_k$ in $C_0$. It can be shown that $\theta_k(\zeta)$ maps $C_k'$ onto $C_k$.

Let $\Theta$ denote the set of all compositions of the maps $\{\theta_k(\zeta) \mid k = 1, \ldots, M\}$ and their inverses. It is an example of an infinite *Schottky group*. Further information on Schottky groups can be found in [3], [4]. We refer to the maps $\{\theta_k(\zeta) \mid k = 1, \ldots, M\}$, together with their inverses, as the *generators* of $\Theta$. A *fundamental region* of $\Theta$ is a connected region whose images under all maps in $\Theta$ tessellate the whole of the plane. Consider the region consisting of $D_\zeta$ and its reflection in $C_0$, i.e., the $2M$-connected region bounded by $\{C_k, C_k' \mid k = 1, \ldots, M\}$. Label this region as $F$. $F$ is a fundamental region of $\Theta$.

Associated with $\Theta$ are $M$ functions known as *integrals of the first kind* which we denote $\{\upsilon_k(\zeta) \,|\; k = 1, \ldots, M\}$. These are analytic, but not single-valued, in $F$. Indeed, for $j, k = 1, \ldots, M$ we have

$$[\upsilon_k(\zeta)]_{C_j} = -[\upsilon_k(\zeta)]_{C'_j} = \delta_{jk}, \tag{4}$$

where $[\upsilon_k(\zeta)]_{C_j}$ and $[\upsilon_k(\zeta)]_{C'_j}$ denote respectively the changes in $\upsilon_k(\zeta)$ on traversing $C_j$ and $C'_j$ with the interior of $F$ on the right, and $\delta_{jk}$ denotes the Kronecker delta function. Furthermore, for $j, k = 1, \ldots, M$,

$$\upsilon_k(\theta_j(\zeta)) - \upsilon_k(\zeta) = \tau_{jk} \tag{5}$$

for some $\{\tau_{jk} \,|\; j, k = 1, \ldots, M\}$ which are constants, i.e., independent of $\zeta$. The functions $\{\upsilon_k(\zeta) \,|\; k = 1, \ldots, M\}$ are uniquely determined (up to an additive constant) by their periods given by (4) and (5).

## 2.1. The Schottky–Klein prime function

Let $\alpha$ be some arbitrary point in $F$. It is established in [12] that there exists a unique function $X(\zeta, \alpha)$ defined by the properties:

(i) $X(\zeta, \alpha)$ is single-valued and analytic in $F$.

(ii) $X(\zeta, \alpha)$ has a second-order zero at each of the points $\theta(\alpha)$, $\theta \in \Theta$.

(iii) $\lim_{\zeta \to \alpha} \frac{X(\zeta,\alpha)}{(\zeta-\alpha)^2} = 1$.

(iv) For $k = 1, \ldots, M$,

$$X(\theta_k(\zeta), \alpha) = \exp\left(-2\pi \mathrm{i}(2\upsilon_k(\zeta) - 2\upsilon_k(\alpha) + \tau_{kk})\right) \frac{d\theta_k(\zeta)}{d\zeta} X(\zeta, \alpha).$$

The *Schottky–Klein prime function* (henceforth referred to as S–K prime function), which we denote $\omega(\zeta, \alpha)$, is defined as

$$\omega(\zeta, \alpha) = (X(\zeta, \alpha))^{1/2},$$

where the branch of the square root is chosen so that $\omega(\zeta, \alpha)$ behaves like $(\zeta - \alpha)$ as $\zeta \to \alpha$.

There are two known ways to evaluate the S–K prime function. One possibility is to use a classical infinite product formula for it as recorded, for example, in Baker [3]. It is given by

$$\omega(\zeta, \alpha) = (\zeta - \alpha) \prod_{\theta_k} \frac{(\theta_k(\zeta) - \alpha)(\theta_k(\alpha) - \zeta)}{(\theta_k(\zeta) - \zeta)(\theta_k(\alpha) - \alpha)}, \tag{6}$$

where the product is over all compositions of the basic maps $\{\theta_j, \theta_j^{-1} \,|\; j = 1, \ldots, M\}$ excluding the identity and all inverse maps. This product, even if it is convergent, can converge so slowly and require such a large number of terms in the product, that its use in many circumstances is impractical. An alternative numerical scheme has recently been put forward by Crowdy & Marshall [8]; it is much more computationally efficient than methods based on the infinite product (6) over the Schottky group.

## 3. The circular slit domain

To proceed with the construction, we introduce an intermediate $\eta$-plane. Consider a conformal mapping, denoted $\eta(\zeta;\alpha)$, taking the multiply connected circular domain $D_\zeta$ to a conformally equivalent circular slit domain called $D_\eta$. $\alpha$ is the point in $D_\zeta$ mapping to $\eta = 0$ in $D_\eta$, i.e., $\eta(\alpha;\alpha) = 0$. Figure 1 shows a schematic in a triply connected case. Let the image of $C_0$ under this mapping be the unit circle in the $\eta$-plane which will be called $L_0$. The $M$ circles $\{C_j \mid j = 1,\ldots,M\}$ will be taken to have circular-slit images, centred on $\eta = 0$, and labelled $\{L_j \mid j = 1,\ldots,M\}$. Let the circular arc $L_j$ be characterized by the conditions

$$|\eta| = r_j, \qquad \arg[\eta] \in [\phi_1^{(j)}, \phi_2^{(j)}].$$

There will be two pre-image points on the circle $C_j$ corresponding to the two end-points of the circular-slit $L_j$. These two pre-image points, labelled $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$, satisfy the conditions

$$\eta(\gamma_1^{(j)};\alpha) = r_j e^{i\phi_1^{(j)}}, \qquad \eta_\zeta(\gamma_1^{(j)},\alpha) = 0,$$
$$\eta(\gamma_2^{(j)};\alpha) = r_j e^{i\phi_2^{(j)}}, \qquad \eta_\zeta(\gamma_2^{(j)},\alpha) = 0.$$

These two zeros of $\eta_\zeta(\zeta)$ on $C_j$ are simple zeros.

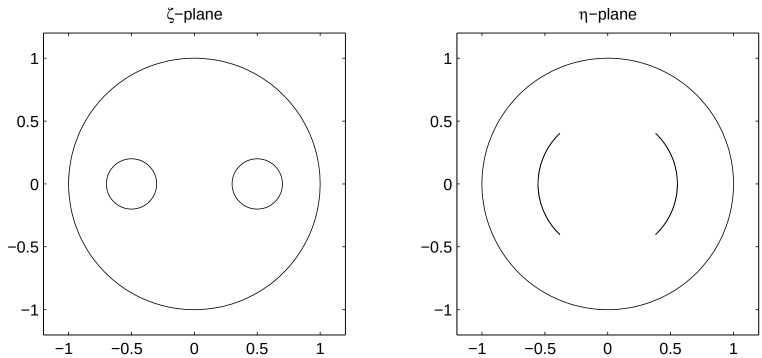


Figure 1: A typical circular slit mapping from a triply connected circular region $D_\zeta$ in a $\zeta$-plane to a triply connected circular slit domain $D_\eta$ in a $\eta$-plane.

It is shown in [5] and [7] that an explicit expression for the conformal slit mapping from $D_\zeta$ to $D_\eta$ can be found in terms of the S–K prime function of $D_\zeta$. It is given by

$$\eta(\zeta;\alpha) = \frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\overline{\alpha}^{-1})}. \tag{7}$$

Formula (7) will be crucial in the solution scheme to follow.

## 4. Solution scheme

The required function $w(\zeta)$ is analytic in $D_\zeta$. One can also consider the composed function $W(\eta)$, analytic in $D_\eta$, defined by

$$W(\eta(\zeta;\alpha)) \equiv w(\zeta).$$

The boundary conditions (1), expressed in terms of this new function $W(\eta)$, are

$$\mathrm{Re}\left[\overline{\lambda_k}\, W(\eta)\right] = d_k \qquad \text{on } L_k,\ k = 0, 1, \ldots, M.$$

These can be rewritten in the form

$$\overline{\lambda_k}\, W(\eta) + \lambda_k \overline{W(\eta)} = 2d_k \qquad \text{on } L_k,\ k = 0, 1, \ldots, M,$$

or, on use of the fact that $\overline{\eta} = r_k^2 \eta^{-1}$ on $L_k$,

$$\overline{\lambda_k}\, W(\eta) + \lambda_k \overline{W}(r_k^2 \eta^{-1}) = 2d_k \qquad \text{on } L_k,\ k = 0, 1, \ldots, M. \tag{8}$$

Using $W'(\eta)$ to denote the derivative of $W$ with respect to its argument, differentiation of (8) with respect to $\eta$ gives

$$\overline{\lambda_k}\, W'(\eta) - \frac{r_k^2}{\eta^2} \lambda_k \overline{W}'(r_k^2 \eta^{-1}) = 0 \qquad \text{on } L_k,\ k = 0, 1, \ldots, M,$$

which can be rewritten as

$$\frac{\eta W'(\eta)}{\overline{\eta W'(\eta)}} = \frac{\lambda_k}{\overline{\lambda_k}} \qquad \text{on } L_k,\ k = 0, 1, \ldots, M.$$

This is a statement of the fact that the argument of $\eta W'(\eta)$ is constant on $L_k$.

Let us now suppose that we seek a solution for which there are precisely two zeros of the derivative $dw/d\zeta$ on each of the boundary components $\{C_j \mid j = 0, 1, \ldots, M\}$. Let the positions of the two zeros on $C_j$ be at points $a_j$ and $c_j$, i.e.,

$$\frac{dw}{d\zeta}(a_j) = 0 = \frac{dw}{d\zeta}(c_j).$$

These zero positions will not be known *a priori* but will enter our representation of the solution as accessory parameters.

### 4.1. Building block functions

A set of "building block" functions will be used to construct the required solutions. Their characterizing feature is that they all have constant argument on the boundary circles $\{C_j \mid j = 0, 1, \ldots, M\}$. These functions were introduced in [5] and their properties established there.

It is shown in [5] that functions of the form

$$R_1(\zeta; \zeta_1, \zeta_2) = \frac{\omega(\zeta, \zeta_1)}{\omega(\zeta, \zeta_2)}, \tag{9}$$

where $\zeta_1$ and $\zeta_2$ are any two points on the *same* circle $C_k$ (for $k = 0, 1, \ldots, M$) has constant argument on each of the boundary circles $\{C_j \mid j = 0, 1, \ldots, M\}$. Also, functions of the form

$$R_2(\zeta; \zeta_1, \zeta_2) = \frac{\omega(\zeta, \zeta_1)\omega(\zeta, \overline{\zeta_1}^{-1})}{\omega(\zeta, \zeta_2)\omega(\zeta, \overline{\zeta_2}^{-1})}, \tag{10}$$

where $\zeta_1$ and $\zeta_2$ are any two ordinary points of the Schottky group (these points need not be points on the boundary circles) similarly have constant argument on each of the boundary circles $\{C_j \mid j = 0, 1, \ldots, M\}$.

Let $\gamma_0$ be some point on $C_0$ that is distinct from $a_0$ and $c_0$. Consider the function

$$R_1(\zeta; a_0, \gamma_0)R_1(\zeta; c_0, \gamma_0)R_2(\zeta; \gamma_0, \beta)R_2(\zeta; \alpha, \beta) \times \prod_{k=1}^{M} R_1(\zeta; a_k, \gamma_k^{(1)})R_1(\zeta; c_k, \gamma_k^{(2)}). \tag{11}$$

First, since it is a product of the building block functions just introduced, the function in (11) has constant argument on the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. As for its singularities, it is a meromorphic function in $D_\zeta$ with a second order pole at $\zeta = \beta$ (and at $\overline{\beta}^{-1}$), simple poles at the points $\{\gamma_k^{(1)}, \gamma_k^{(2)} \mid k = 1, \ldots, M\}$, simple zeros at $\zeta = \alpha$ and $\overline{\alpha}^{-1}$ and simple zeros at the points $\{a_k, c_k \mid k = 0, 1, \ldots, M\}$. It has no other singularities in $D_\zeta$. Let the function (11), considered now as a function of $\eta$, be called $U(\eta)$.

Now consider the function $\eta W'(\eta)$ which, we have already established, must have constant argument on the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. By the chain rule we have

$$\eta W'(\eta) = \eta \frac{dw/d\zeta}{d\eta/d\zeta}.$$

This function is analytic everywhere in $D_\eta$ except for simple poles at the zeros of $d\eta/d\zeta$, i.e., at the points $\{\gamma_k^{(1)}, \gamma_k^{(2)} \mid k = 1, \ldots, M\}$. It also has second order poles at $\zeta = \beta$ and $\overline{\beta}^{-1}$. It has a simple zero at $\zeta = \alpha$ since $\eta(\zeta; \alpha)$ has a simple zero there and, as can be seen after making use of (7), it also has a simple zero at $\overline{\alpha}^{-1}$. By assumption, it also has $2(M+1)$ simples zeros at the points $\{a_k, c_k \mid k = 0, 1, \ldots, M\}$. In short, it has all the same zeros and poles in $D_\zeta$ as the function $U(\eta)$.

We are thus led to consider the ratio

$$V(\eta) \equiv \frac{\eta W'(\eta)}{U(\eta)}$$

in the domain $D_\eta$. Since we know that $U(\eta)$ and $\eta W'(\eta)$ have the same poles and zeros inside and on the boundaries of $D_\zeta$, the function $V(\eta)$ can be deduced to be analytic everywhere in the domain $D_\eta$, as well as on its boundaries. This means that $V(\eta)$ is analytic everywhere in $|\eta| \le 1$. Moreover, it is known that the arguments of both $U(\eta)$ and $\eta W'(\eta)$ are constant on $L_0$. Thus,

$$\overline{V(\eta)} = \epsilon V(\eta) \qquad \text{on } L_0,$$

for some constant $\epsilon$ implying that

$$\overline{V}(\eta^{-1}) = \epsilon V(\eta) \qquad \text{on } L_0.$$

This equation furnishes the analytic continuation of $V(\eta)$ into $|\eta| > 1$ and, in particular, shows that it is analytic there (and bounded at infinity). Since $V(\eta)$ is analytic everywhere in the complex $\eta$-plane, and bounded as $\eta \to \infty$, Liouville's theorem implies $V(\eta) = B$, where $B$ is some complex constant.

On use of (9) and (10), and after some cancellations, we deduce that

$$\frac{dw(\zeta)}{d\zeta} = \frac{BS(\zeta;\alpha)}{\omega(\zeta,\beta)^2\omega(\zeta,\overline{\beta}^{-1})^2} \prod_{k=0}^{M} \omega(\zeta,a_k)\omega(\zeta,c_k),$$

where

$$S(\zeta;\alpha) \equiv \left( \frac{\omega(\zeta,\overline{\alpha}^{-1})\omega_\zeta(\zeta,\alpha) - \omega(\zeta,\alpha)\omega_\zeta(\zeta,\overline{\alpha}^{-1})}{\prod_{k=1}^{M} \omega(\zeta,\gamma_k^{(1)})\omega(\zeta,\gamma_k^{(2)})} \right).$$

Hence, the required solution can be written as the indefinite integral

$$w(\zeta) = A + B \int_1^\zeta \frac{S(\zeta';\alpha)}{\omega(\zeta',\beta)^2\omega(\zeta',\overline{\beta}^{-1})^2} \prod_{k=0}^{M} \omega(\zeta',a_k)\omega(\zeta',c_k)\, d\zeta', \tag{12}$$

where $A$ is some complex constant. Formula (12) is the main result of this paper.

It is demonstrated in the appendix that for any two distinct choices of $\alpha_1$ and $\alpha_2$, $S(\zeta;\alpha_1) = CS(\zeta;\alpha_2)$, where $C$ is some constant (independent of $\zeta$). This means that making different choices of $\alpha$ in the representation (12) simply corresponds to making a different choice of the constant $B$.

## 5. The doubly connected case

As verification we consider two problems in the doubly connected case. Let $D_\zeta$ to be the concentric annulus $\rho < |\zeta| < 1$ for some real $\rho$. Any doubly connected domain is conformally equivalent to some such annulus. The solutions to the following two problems can, it turns out, be found in analytical form using alternative arguments which allows us to check our analysis.

Problem 1

We specialize to the case where $\lambda_0 = \lambda_1 = 1$ with $c_0 = 0$. The problem is then the classical Schwarz problem. One form of the solution is

$$w(\zeta) = \frac{U}{\zeta - \beta} + \tilde{A}\log\zeta + I(\zeta), \tag{13}$$

where $\tilde{A}$ is a constant and the single-valued function $I(\zeta)$ can be written in terms of the classical Villat formula [1]:

$$I(\zeta) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} (1 - 2K(\zeta/\zeta',\rho)) \left[ -\mathrm{Re}\left[ \frac{U}{\zeta - \beta} + \tilde{A}\log\zeta \right] \right]$$

$$-\frac{1}{2\pi i}\oint\limits_{|\zeta'|=\rho}\frac{d\zeta'}{\zeta'}\left(2-2K(\zeta/\zeta',\rho)\right)\left[c_1-\mathrm{Re}\left[\frac{U}{\zeta-\beta}+\tilde{A}\log\zeta\right]\right],$$

where

$$K(\zeta,\rho)\equiv\frac{\zeta P_\zeta(\zeta,\rho)}{P(\zeta,\rho)} \tag{14}$$

and

$$P(\zeta,\rho)\equiv(1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{15}$$

Alternatively, the same solution can be written in the form

$$w(\zeta)=\frac{U}{\beta}\left(K(\zeta\beta^{-1},\rho)+K(\zeta\overline{\beta},\rho)\right)+C\log\zeta+D,$$

where

$$C=\frac{1}{\log\rho}\left(c_1-\frac{U}{\beta}\left(K(\rho\beta^{-1},\rho)+K(\rho\overline{\beta},\rho)-K(\beta^{-1},\rho)-K(\overline{\beta},\rho)\right)\right)$$

and

$$D=-\frac{U}{\alpha}\left(K(\beta^{-1},\rho)+K(\overline{\beta},\rho)\right).$$

The new solution method given earlier provides a third representation of the same solution:

$$w(\zeta)=A+B\int_1^{\zeta}\frac{S(\zeta';\alpha)}{\omega(\zeta',\beta)^2\omega(\zeta',\overline{\beta}^{-1})^2}\omega(\zeta',a_0)\omega(\zeta',c_0)\omega(\zeta',a_1)\omega(\zeta',c_1)\,d\zeta', \tag{16}$$

where, in this doubly connected case, it can be shown (see [5] for details) that

$$S(\zeta;\alpha)\propto\frac{1}{\zeta^2}.$$

To check (16) we use the solution (13) to numerically compute (using Newton's method) the two points on $C_0$ at which $dw/d\zeta=0$. These are substituted into (16) as the values of $a_0$ and $c_0$. Similarly, we find the two points on $C_1$ at which $dw/d\zeta=0$ and take these as the values of $a_1$ and $c_1$. Next, we set $A=w(1)$, where the right hand side is computed using the known solution (13). We also fix $B$ by ensuring that

$$w(\rho)=A+B\int_1^{\rho}\frac{S(\zeta')}{\omega(\zeta',\beta)^2\omega(\zeta',\overline{\beta}^{-1})^2}\omega(\zeta',a_0)\omega(\zeta',c_0)\omega(\zeta',a_1)\omega(\zeta',c_1)\,d\zeta',$$

where the left hand side of this equation is evaluated using the known solution (13). With all the parameters in (16) now determined, we check the value of the integral (16) against the values given by (13) for different (arbitrary) choices of $\zeta$ in the annulus (the integral (16) is computed using the trapezoidal rule). The values are found to be in agreement (to within the accuracy of the numerical method) thereby confirming that (16) is indeed a representation of the required solution.

Problem 2

We now specialize to the case where $\lambda_0 = 1$, $\lambda_1 = e^{i\pi/2}$ with no restrictions on $d_0$ and $d_1$. This is no longer a classical Schwarz problem so the Villat formula cannot be used here. An analytical formula for the solution can, however, be found:

$$w(\zeta) = \frac{U}{\beta}\left[K(\zeta\beta^{-1},\rho^2) - K(\zeta\beta,\rho^2) - K(\zeta\beta^{-1}\rho^{-2},\rho^2) + K(\zeta\beta\rho^{-2},\rho^2)\right] + d_0 + id_1, \tag{17}$$

where the special function defined in (14) again appears. A derivation of (17) is given in appendix B. In a manner akin to that used in Problem 1, the expression (17) was used to find the locations of the zeros of $dw/d\zeta$ on both $C_0$ and $C_1$ (there are two on each circle). These are then used as the values of $a_0$, $c_0$, $a_1$ and $c_1$ in an expression of the form (16). The values of $A$ and $B$ are determined in the same way as in Problem 1 and the values of the integral (16) for arbitrary values of $\zeta$ checked against the values given by (17). They are found to be in agreement.

## 6. Discussion

This paper describes a constructive method for finding solutions to Riemann–Hilbert problems of the special form (1) on multiply connected domains. The solution having two zeros of the derivative on each of the boundary circles is given in (12) as a non-singular indefinite integral containing a finite set of accessory parameters. In general, these parameters must be determined from a set of equations obtained by substituting the form (12) into the boundary conditions (1). In other words, given the $2M+2$ real parameters associated with the set $\{\lambda_k, d_k \mid k = 0, 1, \ldots, M\}$ it is possible to determine the $2M+2$ real parameters associated with the set of zeros $\{a_k, c_k \mid k = 0, 1, \ldots, M\}$. How to determine these accessory parameters numerically in an efficient manner remains a subject for future research.

In principle, it is possible to extend the constructive method herein to find representations to solutions of the discontinuous analogues of the special RH problems considered here where the constant $\lambda_k$ is allowed to assume different piecewise constant values on different segments of circle $C_k$. In such cases, one must generally introduce branch point singularities in the derivative $w_\zeta(\zeta)$ but this just requires the incorporation of appropriate non-integer powers of the building block functions when performing the construction described herein. It is very similar to what is done in constructing multiply connected Schwarz–Christoffel formulae [5], [6].

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## A. The function $S(\zeta;\alpha)$

In this appendix we establish the fact that $S(\zeta;\alpha_1) = CS(\zeta;\alpha_2)$, where $C$ is some constant (independent of $\zeta$). To this end, consider the ratio

$$R(\zeta) \equiv \frac{S(\zeta;\alpha_1)}{S(\zeta;\alpha_2)}, \tag{18}$$

where $\alpha_1$ and $\alpha_2$ are two distinct values in $D_\zeta$. First, notice that $S(\zeta;\alpha_1)$ can be rewritten in the form

$$S(\zeta;\alpha_1) = \left(\frac{\omega_\zeta(\zeta,\alpha_1)}{\omega(\zeta,\alpha_1)} - \frac{\omega_\zeta(\zeta,\overline{\alpha_1}^{-1})}{\omega(\zeta,\overline{\alpha_1}^{-1})}\right)\frac{\omega(\zeta,\alpha_1)\omega(\zeta,\overline{\alpha_1}^{-1})}{\prod_{k=1}^{M}\omega(\zeta,\gamma_k^{(1)})\omega(\zeta,\gamma_k^{(2)})}, \tag{19}$$

where $\{\gamma_k^{(1)},\gamma_k^{(2)} \mid k=1,\ldots,M\}$ are the zeros of the slit map $\eta(\zeta;\alpha_1)$. Similarly

$$S(\zeta;\alpha_2) = \left(\frac{\omega_\zeta(\zeta,\alpha_2)}{\omega(\zeta,\alpha_2)} - \frac{\omega_\zeta(\zeta,\overline{\alpha_2}^{-1})}{\omega(\zeta,\overline{\alpha_2}^{-1})}\right)\frac{\omega(\zeta,\alpha_2)\omega(\zeta,\overline{\alpha_2}^{-1})}{\prod_{k=1}^{M}\omega(\zeta,\tilde{\gamma}_k^{(1)})\omega(\zeta,\tilde{\gamma}_k^{(2)})}, \tag{20}$$

where $\{\tilde{\gamma}_k^{(1)},\tilde{\gamma}_k^{(2)} \mid k=1,\ldots,M\}$ are the zeros of the slit map $\eta(\zeta;\alpha_2)$. It is also easy to check that, for $j=1,2$,

$$\frac{\omega_\zeta(\zeta,\alpha_j)}{\omega(\zeta,\alpha_j)} - \frac{\omega_\zeta(\zeta,\overline{\alpha_j}^{-1})}{\omega(\zeta,\overline{\alpha_j}^{-1})} = \frac{\eta_\zeta(\zeta;\alpha_j)}{\eta(\zeta;\alpha_j)}.$$

Next, observe that on $C_j$ (and for *any* $\alpha$),

$$\eta(\zeta;\alpha)\overline{\eta(\zeta;\alpha)} = r_j^2,$$

where $r_j$ is some real constant. A differentiation with respect to $\zeta$ yields

$$\frac{\eta_\zeta(\zeta;\alpha)}{\eta(\zeta;\alpha)} = -\left(\frac{d\overline{\zeta}}{d\zeta}\right)\overline{\left(\frac{\eta_\zeta(\zeta;\alpha)}{\eta(\zeta;\alpha)}\right)}.$$

It follows that the *ratio* of any two such functions, that is,

$$T(\zeta) \equiv \frac{\eta_\zeta(\zeta;\alpha_1)/\eta(\zeta;\alpha_1)}{\eta_\zeta(\zeta;\alpha_2)/\eta(\zeta;\alpha_2)}$$

will be real (and, in particular, have constant argument) on all the circles $\{C_j \mid j = 0,1,\ldots,M\}$.

Substitution of (19) and (20) into (18) then produces

$$R(\zeta) = T(\zeta)R_2(\zeta;\alpha_1,\alpha_2)\prod_{j=1}^{M} R_1(\zeta;\tilde{\gamma}_j^{(1)},\gamma_j^{(1)})R_1(\zeta;\tilde{\gamma}_j^{(2)},\gamma_j^{(2)}).$$

The important observation is that this is a product of functions that all have constant argument on the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. These conditions can be written as

$$\overline{R(\zeta)} = \kappa_j R(\zeta) \qquad \text{on } C_j,\ j = 0, 1, \ldots, M, \tag{21}$$

for some set of complex constants $\{\kappa_j \mid j = 0, 1, \ldots, M\}$. $R(\zeta)$ can be shown to be a constant. One way to do this is to use arguments similar to those used in §4.1. to show that $V(\eta)$ is constant, but it is instructive to present an alternative argument based on RH methods. The function $R(\zeta)$ is known to be analytic and single-valued everywhere in the fundamental region of the group $\Theta$. Consider the real part of equation (21); it can be written in the standard form of a RH problem:

$$\mathrm{Re}[\overline{\mu_j}\, R(\zeta)] = 0 \qquad \text{on } C_j,\ j = 0, 1, \ldots, M, \tag{22}$$

for some set of complex constants $\{\mu_j \mid j = 0, 1, \ldots, M\}$. The (homogeneous) Riemann–Hilbert problem (22) has been well studied and it is known (see, for example, p. 257 of Vekua [18]) that it admits no solution for $R(\zeta)$ unless all the constants $\{\mu_j \mid j = 0, 1, \ldots, M\}$ are identical. In this case, the unique solution is $R(\zeta) = C$, where $C$ is a constant. Thus, we have established that $S(\zeta; \alpha_1) = CS(\zeta; \alpha_2)$ for some constant $C$ that is independent of $\zeta$.

## B. Derivation of (17)

To find solution (17), consider the following boundary value problem for $w(\zeta)$:

$$\begin{aligned} \mathrm{Re}[w(\zeta)] &= 0 \qquad \text{on } |\zeta| = 1, \\ \mathrm{Im}[w(\zeta)] &= 0 \qquad \text{on } |\zeta| = \rho. \end{aligned}$$

These imply that

$$\begin{aligned} w(\zeta) + \overline{w}(\zeta^{-1}) &= 0 \qquad \text{on } |\zeta| = 1, \\ w(\zeta) - \overline{w}(\rho^2\zeta^{-1}) &= 0 \qquad \text{on } |\zeta| = \rho. \end{aligned} \tag{23}$$

The relations (23) can be analytically continued off the respective circles and imply that $w(\zeta)$ satisfies the functional relation

$$w(\rho^4\zeta) = w(\zeta). \tag{24}$$

Now $P(\zeta, \rho)$ can be shown, directly from its definition (15), to satisfy the functional relations

$$P(\zeta^{-1}, \rho) = -\zeta^{-1} P(\zeta, \rho), \qquad P(\rho^2\zeta, \rho) = -\zeta^{-1} P(\zeta, \rho),$$

from which it also follows that

$$K(\zeta^{-1}, \rho) = 1 - K(\zeta, \rho), \qquad K(\rho^2\zeta, \rho) = K(\zeta, \rho) - 1.$$

Furthermore, near $\zeta = 1$, $K(\zeta, \rho)$ has a simple pole with unit residue, i.e.,

$$K(\zeta, \rho) = \frac{1}{\zeta - 1} + \text{analytic}.$$

We can therefore use $K(\zeta,\rho^2)$ to construct a function $w(\zeta)$ satisfying (24) and having a simple pole at $\zeta=\beta$. The relations (23) imply that $w(\zeta)$ also has simple poles at $\zeta=\beta^{-1},\rho^2\beta,\rho^2\beta^{-1}$ (and at all points equivalent to these under $\zeta\mapsto\rho^4\zeta$). The required form of solution can now easily be deduced to be that given in (17).

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# On some equations stemming from quadrature rules

**Abstract.** We deal with functional equations of the type

$$F(y) - F(x) = (y - x) \sum_{k=1}^{n} f_k \left( (1 - \lambda_k)x + \lambda_k y \right),$$

connected to quadrature rules and, in particular, we find the solutions of the following functional equation

$$f(x) - f(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)].$$

We also present a solution of the Stamate type equation

$$yf(x) - xf(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)].$$

All results are valid for functions acting on integral domains.

## 1. Introduction

We deal with some equations connected to quadrature rules. Having a function $f\colon \mathbb{R} \to \mathbb{R}$ we may approximate its integral using the following expression

$$F(y) - F(x) \approx (y - x) \sum_{k=1}^{n} \alpha_k f((1 - \lambda_k)x + \lambda_k y)$$

(where $F$ is a primitive function for $f$), which is satisfied exactly for polynomials of certain degree. One of the simplest functional equations connected to quadrature rules is an equation stemming from Simpson's rule

$$F(y) - F(x) = (y - x) \left[ \frac{1}{6} f(x) + \frac{2}{3} f\left( \frac{x + y}{2} \right) + \frac{1}{6} f(y) \right].$$

Another example is given by the equation

$$F(y) - F(x) = (y - x) \left[ \frac{1}{8} f(x) + \frac{3}{8} f\left( \frac{x + 2y}{3} \right) + \frac{3}{8} f\left( \frac{2x + y}{3} \right) + \frac{1}{8} f(y) \right],$$

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which is satisfied by polynomials of degree not greater than 3. The generalized version of this equation

$$g(x) - f(y) = (x - y)[h(x) + k(sx + ty) + k(tx + sy) + h(y)] \tag{1}$$

was considered during the 44th ISFE held in Louisville, Kentucky, USA by P.K. Sahoo [7]. The solution has been given in the class of functions $f$, $g$, $h$, $k$ mapping $\mathbb{R}$ into $\mathbb{R}$ and such that $g$ and $f$ are twice differentiable, and $k$ is four times differentiable.

On the other hand, M. Sablik [5] during the *7th* Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities presented the general solution of this equation in the case $s, t \in \mathbb{Q}$ without any regularity assumptions concerning the functions considered.

We deal with a special case of (1) (with $s = 1$, $t = 2$) for functions acting on integral domains. However, it is easy to observe that if we take $x = y$ in (1), then we immediately obtain that $f = g$. Thus we shall find the solutions of the following functional equation

$$f(x) - f(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)]. \tag{2}$$

Using the obtained result we will also present a solution of a similar Stamate type equation

$$yf(x) - xf(y) = (x - y)[g(x) + h(2x + y) + h(x + 2y) + g(y)]. \tag{3}$$

In the proof of Lemma 1 below we use the lemma established by M. Sablik [6] and improved by I. Pawlikowska [3]. First we need some notations. Let $G$, $H$ be Abelian groups and $SA^0(G,H) := H$, $SA^1(G,H) := \mathrm{Hom}(G,H)$ (i.e., the group of all homomorphisms from $G$ into $H$), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G,H)$ be the group of all $i$-additive and symmetric mappings from $G^i$ into $H$. Furthermore, let $\mathcal{P} := \{(\alpha, \beta) \in \mathrm{Hom}(G,G)^2 : \alpha(G) \subset \beta(G)\}$. Finally, for $x \in G$ let $x^i = (\underbrace{x, \ldots, x}_{i})$, $i \in \mathbb{N}$.

Lemma 1

*Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0, \ldots, I_N$ be finite subsets of $\mathcal{P}$. Suppose that $H$ is uniquely divisible by $N!$ and let the functions $\varphi_i \colon G \to SA^i(G,H)$ and $\psi_{i,(\alpha,\beta)} \colon G \to SA^i(G,H)$ $((\alpha, \beta) \in I_i$, $i = 0, \ldots, N)$ satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(y^i)$$

*for every $x, y \in G$. Then $\varphi_N$ is a polynomial function of order at most $k - 1$, where*

$$k = \sum_{i=0}^{N} \mathrm{card}\left(\bigcup_{s=i}^{N} I_s\right).$$

Now we will state a simplified version of this lemma. We take $N = 1$ and we consider functions acting on an integral domain $P$. Moreover, we consider only homomorphisms of the type $x \mapsto yx$, where $y \in P$ is fixed.

Lemma 2
*Let $P$ be an integral domain and let $I_0$, $I_1$ be finite subsets of $P^2$ such that for all $(a, b) \in I_i$ the ring $P$ is divisible by $b$. Let $\varphi_i, \psi_{i,(\alpha,\beta)}\colon P \to P$ satisfy*

$$\varphi_1(x)y + \varphi_0(x) = \sum_{(a,b)\in I_0} \psi_{0,(a,b)}(ax + by) + y \sum_{(a,b)\in I_1} \psi_{1,(a,b)}(ax + by)$$

*for all $x, y \in P$. Then $\varphi_1$ is a polynomial function of order at most equal to* $\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1$.

In the above lemmas a *polynomial function of order* $n$ means a solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where $\Delta_h^n$ stands for the $n$-th iterate of the difference operator $\Delta_h f(x) = f(x + h) - f(x)$. Observe that a continuous polynomial function of order $n$ is a polynomial of degree at most $n$ (see [2, Theorem 4, p. 398]).

It is also well known that if $P$ is an integral domain uniquely divisible by $n!$ and $f\colon P \to P$ is a polynomial function of order $n$, then

$$f(x) = c_0 + c_1(x) + \ldots + c_n(x), \qquad x \in P,$$

where $c_0 \in P$ is a constant and

$$c_i(x) = C_i(x, x, \ldots, x), \qquad x \in P$$

for some $i$-additive and symmetric function $C_i\colon P^i \to P$.

## 2. Results

We begin with the following lemma which will be usefull in the proof of the main result. However, we state it a bit more generally.

Lemma 3
*Let $P$ be an integral domain and let $f, f_k\colon P \to P$, $k = 0, \ldots, n$, be functions satisfying the equation*

$$f(y) - f(x) = (y - x) \sum_{k=0}^{n} f_k(a_k x + b_k y), \tag{4}$$

*where $a_k, b_k \in P$ are given numbers such that for every $k \in \{0, \ldots, n\}$ we have $a_k \neq 0$ or $b_k \neq 0$.*

*Let $i \in \{0, \ldots, n\}$ be fixed. If $P$ is divisible by $a_i$, $b_i$ and also by $a_i b_k - a_k b_i$, $k = 0, \ldots, n$; $k \neq i$, then the function*

$$\tilde{f}(x) := (a_i + b_i) f_i((a_i + b_i)x)$$

*is a polynomial function of degree at most $2n + 1$.*

*Moreover, if there exists $k_1 \in \{0, 1, \dots, n\}$ such that $a_{k_1} = 0$ or $b_{k_1} = 0$, then function $\tilde{f}$ is a polynomial function of order at most $2n$ and if there exist $k_1, k_2 \in \{0, \dots, n\}$ such that $a_{k_1} = b_{k_2} = 0$, then $\tilde{f}$ is a polynomial function of order at most $2n - 1$.*

*Proof.* Fix an $i \in \{0, \dots, n\}$, put in (4) $x - b_i y$ and $x + a_i y$ instead of $x$ and $y$, respectively, to obtain

$$\begin{aligned} & f(x + a_i y) - f(x - b_i y) \\ & \quad = (a_i + b_i)y[f_0((a_0 + b_0)x + (a_i b_0 - a_0 b_i)y) + \dots \\ & \qquad + f_i((a_i + b_i)x) + \dots + f_n((a_n + b_n)x + (a_i b_n - a_n b_i)y)]. \end{aligned} \tag{5}$$

There are two possibilities:

1. $a_i, b_i \neq 0$,
2. $a_i = 0$ or $b_i = 0$.

Let us consider the first case. Then from (5) we obtain

$$\begin{aligned} y(a_i + b_i)f_i((a_i + b_i)x) &= f(x + a_i y) - f(x - b_i y) \\ & \quad - (a_i + b_i)y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y), \end{aligned}$$

which means that

$$\begin{aligned} y\tilde{f}(x) &= f(x + a_i y) - f(x - b_i y) \\ & \quad -(a_i + b_i)y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y). \end{aligned} \tag{6}$$

Now we are in position to use Lemma 2 with

$$I_0 = \{(1, -b_i), (1, a_i)\}$$

and

$$I_1 = \{(a_k + b_k, a_i b_k - a_k b_i) : \ k = 0, \dots, n; \ k \neq i\}.$$

We clearly obtain that $\tilde{f}$ is a polynomial function of order at most equal to

$$\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 \leq (n + 2) + n - 1 = 2n + 1.$$

Further, if for example $a_{k_1} = 0$ for some $k_1 \in \{0, \dots, n\}$, $k_1 \neq i$, then we have a summand

$$f_{k_1}(b_{k_1} x + a_i b_{k_1} y) = f_{k_1}(b_{k_1}(x + a_i y))$$

on the right-hand side of (6). Thus we put $\tilde{f}_{k_1}(x) := f_{k_1}(b_{k_1} x)$ and (6) takes form

$$\begin{aligned} & y\tilde{f}(x) \\ & \quad = f(x - b_i y) - f(x + a_i y) \\ & \qquad - (a_i + b_i)y \left[ \sum_{k=0, k\neq i, k_1}^{n} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y) + \tilde{f}_{k_1}(x + a_i y) \right]. \end{aligned}$$

Similarly as before we take

$$I_0 = \{(1, -b_i), (1, a_i)\}$$

and

$$I_1 = \{(a_k + b_k, a_i b_k - a_k b_i) : \ k = 0, \dots, n; \ k \neq i, k_1\} \cup \{(1, a_i)\}.$$

In this case we have $I_0 \cap I_1 = \{(1, a_i)\}$, i.e.,

$$\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 \leq (n+1) + n - 1 = 2n.$$

The proof in the case $a_{k_1} = b_{k_2} = 0$ is similar.

Now we consider the case $a_i = 0$ or $b_i = 0$. Let for example $a_i = 0$, then from (6) we have

$$y(b_i) f_i(b_i x) - f(x) = -f(x - b_i y) - b_i y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x - a_k b_i y),$$

i.e.,

$$y b_i \tilde{f}(x) - f(x) = -f(x - b_i y) - b_i y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x - a_k b_i y).$$

In this case we take

$$I_0 = \{(1, -b_i)\}$$

and

$$I_1 = \{(a_k + b_k, -a_k b_i) : \ k = 0, \dots, n; \ k \neq i\}.$$

Thus similarly as before $\tilde{f}$ is a polynomial function of degree not greater than

$$\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 \leq (n+1) + n - 1 = 2n.$$

It is easy to see that if for some $k_2 \in \{0, \dots, n\}$, $b_{k_2} = 0$, then $\tilde{f}$ is a polynomial function of order at most $2n - 1$.

Now we are in position to state the most important result of this paper. Namely, we give a general solution of (2) for functions acting on integral domains satisfying some assumptions.

Theorem 1

*Let $P$ be an integral domain with unit element $\mathbb{1}$, uniquely divisible by $5!$ and such that for every $n \in \mathbb{N}$ we have $n\mathbb{1} \neq 0$. The functions $f, g, h \colon P \to P$ satisfy the equation (2) if and only if there exist $a, b, c, d, \bar{d}, e \in P$ and an additive function $A \colon P \to P$ such that*

$$\begin{aligned} f(x) &= 18ax^4 + 8bx^3 + cx^2 + 2dx + e, && x \in P,\\ g(x) &= 9ax^3 + 3bx^2 + cx - 3A(x) + d - \bar{d}, && x \in P,\\ h(x) &= ax^3 + bx^2 + A(x) + \bar{d}, && x \in P. \end{aligned}$$

*Proof.* Assume that $f, g, h\colon P \to P$ satisfy the equation (2). From Lemma 3 we know that $g$ and $h$ are polynomial functions of order at most 5. Therefore

$$g(x) = c_0 + c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x), \qquad x \in P \tag{7}$$

and

$$h(x) = d_0 + d_1(x) + d_2(x) + d_3(x) + d_4(x) + d_5(x), \qquad x \in P, \tag{8}$$

where $c_i, d_i\colon P \to P$ are diagonalizations of some $i$-additive and symmetric functions $C_i, D_i\colon P^i \to P$, respectively. Taking in (2) $y = 0$, we obtain the following formula

$$f(x) = x[g(x) + h(x) + h(2x) + g(0)] + f(0), \qquad x \in P, \tag{9}$$

which used in (2) gives us

$$\begin{aligned} &x[g(x) + h(x) + h(2x) + g(0)] - y[g(y) + h(y) + h(2y) + g(0)] \\ &\quad = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)], \qquad x, y \in P. \end{aligned}$$

After some simple calculations we get

$$\begin{aligned} &x[h(2x) + h(x) - h(x + 2y) - h(2x + y) - g_0(y)] \\ &\quad = y[h(2y) + h(y) - h(x + 2y) - h(2x + y) - g_0(x)], \qquad x, y \in P, \end{aligned} \tag{10}$$

where $g_0(x) := g(x) - g(0)$, $x \in P$.

Further, putting $2x$ instead of $y$ in (10), we have

$$h(5x) - h(4x) - h(2x) + h(x) = g_0(2x) - 2g_0(x), \qquad x \neq 0,$$

which is also satisfied for $x = 0$, since $g_0(0) = 0$. Thus

$$h(5x) - h(4x) - h(2x) + h(x) = g_0(2x) - 2g_0(x), \qquad x \in P. \tag{11}$$

By (7) we obtain

$$g_0(2x) - 2g_0(x) = 2c_2(x) + 6c_3(x) + 14c_4(x) + 30c_5(x) \tag{12}$$

and similarly from (8) we have

$$h(5x) - h(4x) - h(2x) + h(x) = 6d_2(x) + 54d_3(x) + 354d_4(x) + 2070d_5(x). \tag{13}$$

Using (13) and (12) in (11) we may write

$$6d_2(x) + 54d_3(x) + 354d_4(x) + 2070d_5(x) = 2c_2(x) + 6c_3(x) + 14c_4(x) + 30c_5(x).$$

Comparing the corresponding terms on both sides of this equality we get

$$\begin{aligned} c_2(x) &= 3d_2(x), \\ c_3(x) &= 9d_3(x), \\ 7c_4(x) &= 177d_4(x), \\ c_5(x) &= 69d_5(x). \end{aligned}$$

Using these equations in (7) we have

$$g(x) = c_0 + c_1(x) + 3d_2(x) + 9d_3(x) + c_4(x) + 69d_5(x), \qquad x \in P, \tag{14}$$

where

$$7c_4(x) = 177d_4(x), \qquad x \in P. \tag{15}$$

Substitute in (10) $-x$ in place of $y$. Then

$$h(2x) + h(-2x) - [h(x) + h(-x)] = g_0(x) + g_0(-x), \qquad x \in P.$$

This, in view of (8) and (14), means that

$$6d_2(x) + 30d_4(x) = 6d_2(x) + 2c_4(x), \qquad x \in P,$$

i.e,

$$c_4(x) = 15d_4(x), \qquad x \in P$$

and from (15) we have

$$d_4(x) = 0, \qquad x \in P \tag{16}$$

and also $c_4 = 0$.

Now we shall show that $d_5(x) = 0$ for all $x \in P$. To this end we put in (10) in places of $x$ and $y$, respectively $-x$ and $2x$. Thus

$$-2h(4x) + 3h(3x) - 2h(2x) - h(-2x) - h(-x) + 3h(0) = -g_0(2x) - 2g_0(-x)$$

for $x \in P$. Similarly as before, using (8), (14) and (16), we have

$$-18d_2(x) - 54d_3(x) - 1350d_5(x) = -18d_2(x) - 54d_3(x) - 2070d_5(x), \qquad x \in P,$$

which means that

$$d_5(x) = 0, \qquad x \in P.$$

Now formulas (14) and (8) take forms

$$g(x) = c_0 + c_1(x) + 3d_2(x) + 9d_3(x), \qquad x \in P \tag{17}$$

and

$$h(x) = d_0 + d_1(x) + d_2(x) + d_3(x), \qquad x \in P. \tag{18}$$

Using these equalities in (10), we get

$$\begin{aligned} x[-c_1(y) - 3d_1(y) + 5d_2(x) - 3d_2(y) - d_2(x+2y) - d_2(2x+y) \\ + 9d_3(x) - 9d_3(y) - d_3(x+2y) - d_3(2x+y)] \\ = y[-c_1(x) - 3d_1(x) + 5d_2(y) - 3d_2(x) - d_2(x+2y) - d_2(2x+y) \\ + 9d_3(y) - 9d_3(x) - d_3(x+2y) - d_3(2x+y)]. \end{aligned}$$

Now, since the ring $P$ is divisible by 3 and 2, the functions $d_i$ are diagonalizations of symmetric and $i$-additive functions $D_i \colon P^i \to P$, i.e., $d_i(x) = D_i(x^i)$, $x \in P$. Using these forms of $d_i$ in the above equation we obtain

$$\begin{aligned} 2(x-y)[4D_2(x,y) + 9D_3(x,x,y) + 9D_3(x,y,y)] \\ = y[c_1(x) + 3d_1(x) + 8d_2(x) + 18d_3(x)] \\ -x[c_1(y) + 3d_1(y) + 8d_2(y) + 18d_3(y)] \end{aligned} \tag{19}$$

for all $x, y \in P$. Put in (19) $-y$ instead of $y$. Then for all $x, y \in P$ we have

$$\begin{aligned} 2(x+y)[-4D_2(x,y) - 9D_3(x,x,y) + 9D_3(x,y,y)] \\ = -y[c_1(x) + 3d_1(x) + 8d_2(x) + 18d_3(x)] \\ -x[-c_1(y) - 3d_1(y) + 8d_2(y) - 18d_3(y)]. \end{aligned} \tag{20}$$

Adding the equations (19) and (20) we arrive at

$$9xD_3(x,y,y) - y[4D_2(x,y) + 9D_3(x,x,y)] = -4xd_2(y), \qquad x, y \in P,$$

and, consequently,

$$9xD_3(x,y,y) - 9yD_3(x,x,y) = 4yD_2(x,y) - 4xd_2(y), \qquad x, y \in P. \tag{21}$$

Interchanging in these equations $x$ with $y$ and using the symmetry of both $D_2$ and $D_3$ we may write

$$9yD_3(x,x,y) - 9xD_3(x,y,y) = 4xD_2(x,y) - 4yd_2(x), \qquad x, y \in P. \tag{22}$$

Now, we add (21) and (22) to get

$$(x+y)D_2(x,y) = xd_2(y) + yd_2(x), \qquad x, y \in P.$$

Put here $x + y$ in place of $x$, then

$$(x+2y)D_2(x+y,y) = (x+y)d_2(y) + yd_2(x+y), \qquad x, y \in P,$$

which yields

$$xD_2(x,y) = yd_2(x), \qquad x, y \in P \tag{23}$$

and changing the roles of $x$ and $y$

$$yD_2(x,y) = xd_2(y), \qquad x, y \in P. \tag{24}$$

Now, we multiply (23) by $y$ and (24) by $x$ to obtain

$$xyD_2(x,y) = y^2 d_2(x), \qquad x, y \in P$$

and

$$xyD_2(x,y) = x^2 d_2(y), \qquad x, y \in P.$$

Thus

$$y^2 d_2(x) = x^2 d_2(y), \qquad x, y \in P,$$

which after substituing $y = \mathbb{1}$ gives the formula

$$d_2(x) = bx^2, \qquad x \in P, \tag{25}$$

where $b := d_2(\mathbb{1})$. Thus from (24) we obtain

$$D_2(x,y) = bxy, \qquad x, y \in P. \tag{26}$$

Using the formulas (25) and (26) in (21) we have

$$yD_3(x,x,y) = xD_3(x,y,y), \qquad x,y \in P. \tag{27}$$

Putting $x+y$ in place of $x$ (27), we get

$$yD_3(x+y,x+y,y) = (x+y)D_3(x+y,y,y),$$

which after some calculations gives

$$yD_3(x,x,y) - (x-y)D_3(x,y,y) = xd_3(y), \qquad x,y \in P.$$

We use here the condition (27). Then

$$xD_3(x,y,y) - (x-y)D_3(x,y,y) = xd_3(y), \qquad x,y \in P,$$

i.e.,

$$yD_3(x,y,y) = xd_3(y), \qquad x,y \in P. \tag{28}$$

Clearly we also have

$$xD_3(x,x,y) = yd_3(x), \qquad x,y \in P. \tag{29}$$

Now, multiply the equation (28) by $x$ and (29) by $y^2$. Then we have

$$xyD_3(x,y,y) = x^2d_3(y), \qquad x,y \in P \tag{30}$$

and

$$xy^2D_3(x,x,y) = y^3d_3(x). \tag{31}$$

On the other hand, we multiply (27) by $y$. We obtain

$$y^2D_3(x,x,y) = xyD_3(x,y,y), \qquad x,y \in P. \tag{32}$$

Using (32) in (30) we arrive at

$$x^2d_3(y) = y^2D_3(x,x,y), \qquad x,y \in P,$$

which multiplied by $x$ yields

$$x^3d_3(y) = xy^2D_3(x,x,y), \qquad x,y \in P. \tag{33}$$

Comparing the equation (31) and (33) we obtain

$$y^3d_3(x) = x^3d_3(y), \qquad x,y \in P,$$

i.e.,

$$d_3(x) = ax^3, \qquad x \in P, \tag{34}$$

where $a := d_3(\mathbb{1})$. Now equalities (28) and (29) take forms

$$D_3(x,y,y) = axy^2, \qquad x,y \in P \tag{35}$$

and

$$D_3(x,x,y) = ax^2y, \qquad x,y \in P. \tag{36}$$

Using the formulas (25), (26), (34), (35) and (36) in (19) we have

$$y[c_1(x) + 3d_1(x)] = x[c_1(y) + 3d_1(y)], \qquad x,y \in P.$$

Substituting here $y = \mathbb{1}$ we obtain

$$c_1(x) + 3d_1(x) = x[c_1(\mathbb{1}) + 3d_1(\mathbb{1})], \qquad x \in P,$$

which means that

$$c_1(x) = cx - 3d_1(x), \qquad x \in P,$$

where $c := c_1(\mathbb{1}) + 3d_1(\mathbb{1})$.

Thus we have shown that the formulas (17) and (18) may be written in the form

$$g(x) = 9ax^3 + 3bx^2 + cx - 3d_1(x) + c_0, \qquad x \in P$$

and

$$h(x) = ax^3 + bx^2 + d_1(x) + d_0, \qquad x \in P,$$

where $d_1$ is a given additive function. Now it suffices to use the obtained expressions in (9), to get the desired formula for $f$.

It is an easy calculation to show that these functions $f$, $g$, $h$ satisfy the equation (2).

With the aid of this theorem we may prove also a Stamate-kind result.

Corollary 1

*Let $P$ be an integral domain with unit element $\mathbb{1}$, uniquely divisible by $5!$ and such that for every $n \in \mathbb{N}$ we have $n\mathbb{1} \neq 0$. Functions $f,g,h\colon P \to P$ satisfy the equation* (3) *if and only if there exist $a,\bar{a},b,c,d,\bar{d} \in P$ and an additive function $A\colon P \to P$ such that*

$$f(x) = \begin{cases} 18ax^3 + 8bx^2 + cx + 2d, & x \neq 0 \\ \bar{a}, & x = 0 \end{cases},$$

$$g(x) = \begin{cases} -9ax^3 - 5bx^2 - 3A(x) - d - \bar{d}, & x \neq 0 \\ d - \bar{d} - \bar{a}, & x = 0 \end{cases},$$

$$h(x) = ax^3 + bx^2 + A(x) + \bar{d}, \qquad x \in P.$$

*Conversely, $f,g,h\colon P \to P$ given by the above equalities satisfy* (2).

*Proof.* First we write the equation (3) in the form

$$\begin{aligned}(y-x)f(y) - yf(y) + (y-x)f(x) + xf(x) \\ = (x-y)[g(x) + h(2x+y) + h(x+2y) + g(y)]\end{aligned}$$

and, consequently,

$$xf(x) - yf(y) = (x-y)[g(x) + f(x) + h(2x+y) + h(x+2y) + g(y) + f(y)].$$

Putting here $k(t) := g(t) + f(t)$ and $F(t) := tf(t)$ for all $t \in P$ we obtain

$$F(x) - F(y) = (x-y)[k(x) + h(2x+y) + h(x+2y) + k(y)], \qquad x, y \in P.$$

Thus, using Theorem 1, we get

$$xf(x) = 18ax^4 + 8bx^3 + cx^2 + 2dx + e, \qquad x \in P, \tag{37}$$

$$g(x) + f(x) = 9ax^3 + 3bx^2 + cx - 3A(x) + d - \bar{d}, \qquad x \in P, \tag{38}$$

$$h(x) = ax^3 + bx^2 + A(x) + \bar{d}, \qquad x \in P.$$

Now, from (37) it easily follows that $e = 0$ and furthermore

$$xf(x) = 18ax^4 + 8bx^3 + cx^2 + 2dx,$$

i.e.,

$$f(x) = 18ax^3 + 8bx^2 + cx + 2d, \qquad x \neq 0,$$

which gives us

$$g(x) = -9ax^3 - 5bx^2 - 3A(x) - d - \bar{d}, \qquad x \neq 0.$$

Moreover, from (38) we get $g(0) + f(0) = d - \bar{d}$, thus putting $\bar{a} := f(0)$ we obtain that $g(0) = d - \bar{d} - \bar{a}$.

On the other hand, it is easy to see that functions given by the above formulae yield a solution of the equation (3).

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# On a multivalued second order differential problem with Jensen multifunction

**Abstract.** The aim of this paper is to present a generalization of the results published in [5] and [8] for continuous Jensen multifunctions. In particular, we study a second order differential problem for multifunctions with the Hukuhara derivative.

Throughout this paper all vector spaces are supposed to be real. Let $X$ be a vector space. We introduce the notations:

$$A + B := \{a + b : \ a \in A, \, b \in B\} \quad \text{and} \quad \lambda A := \{\lambda a : \ a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$.

A subset $K$ of $X$ is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

Let $X$ and $Y$ be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F\colon K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of $Y$, is called *additive* if

$$F(x + y) = F(x) + F(y) \qquad \text{for } x, y \in K$$

and $F$ is *Jensen* if

$$F\left(\frac{x+y}{2}\right) = \frac{F(x) + F(y)}{2} \qquad \text{for } x, y \in K. \tag{1}$$

From now on, we assume that $X$ is a normed vector space, $c(X)$ denotes the family of all compact members of $n(X)$ and $cc(X)$ stands for the family of all convex sets of $c(X)$.

Lemma 1 ([4], Theorem 5.6)
*Let $K$ be a convex cone with zero in $X$ and $Y$ be a topological vector space. A set-valued function $F\colon K \to c(Y)$ satisfies the equation (1) if and only if there exist an additive multifunction $A_F\colon K \to cc(Y)$ and a set $G_F \in cc(Y)$ such that*

$$F(x) = A_F(x) + G_F \qquad \text{for } x \in K.$$

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The *Hukuhara difference* $A - B$ of $A, B \in cc(X)$ is a set $C \in cc(X)$ such that $A = B + C$. By Rådström's Cancellation Lemma [9] it follows that if this difference exists, then it is unique.

For a multifunction $F\colon [a,b] \to cc(X)$ such that there exist the Hukuhara differences $F(t) - F(s)$ as $a \le s \le t \le b$, the *Hukuhara derivative* at $t \in (a,b)$ is defined by the formula

$$DF(t) = \lim_{k \to 0^+} \frac{F(t+k) - F(t)}{k} = \lim_{k \to 0^+} \frac{F(t) - F(t-k)}{k},$$

whenever both these limits exist with respect to the Hausdorff distance $h$ (see [3]). Moreover,

$$DF(a) = \lim_{s \to a^+} \frac{F(s) - F(a)}{s - a}, \qquad DF(b) = \lim_{s \to b^-} \frac{F(b) - F(s)}{b - s}.$$

Let $X$ be a Banach space and let $[a,b] \subset \mathbb{R}$. If a multifunction $F\colon [a,b] \to cc(X)$ is continuous, then there exists the Riemann integral of $F$ (see [3]). We need the following properties of the Riemann integral.

Lemma 2 ([7], Lemma 10)
*If $F\colon [a,b] \to cc(X)$ is continuous, then $H(t) = \int_a^t F(u)\,du$ for $a \le t \le b$ is continuous.*

Lemma 3 ([10], Lemma 4)
*If $F\colon [a,b] \to cc(X)$ is continuous and $H(t) = \int_a^t F(u)\,du$, then $DH(t) = F(t)$ for $a \le t \le b$.*

Let $(K,+)$ be a semigroup. A one-parameter family $\{F_t : t \ge 0\}$ of set-valued functions $F_t\colon K \to n(K)$ is said to be a *cosine family* if

$$F_0(x) = \{x\} \qquad \text{for } x \in K$$

and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) := 2 \bigcup_{y \in F_s(x)} F_t(y)$$

for $x \in K$ and $0 \le s \le t$.

Let $X$ be a normed space. A cosine family is called *regular* if

$$\lim_{t \to 0^+} h(F_t(x), \{x\}) = 0.$$

Example 1
Let $K = [0, +\infty)$ and $F_t(x) = [x \cosh at, x \cosh bt]$, where $0 \le a \le b$. Then $\{F_t : t \ge 0\}$ is a regular cosine family of continuous additive multifunctions.

Example 2
Let $K = [0, +\infty)$ and $F_t(x) = [x, x \cosh t + \cosh t - 1]$. Then $\{F_t : t \ge 0\}$ is a regular cosine family of continuous Jensen multifunctions.

We say that a cosine family $\{F_t : t \geq 0\}$ is *differentiable* if all multifunctions $t \mapsto F_t(x)$ $(x \in K)$ have the Hukuhara derivative on $[0, +\infty)$.

Lemma 4 ([8], Theorem)
*Let $X$ be a Banach space and let $K$ be a closed convex cone with a nonempty interior in $X$. Suppose that $\{A_t : t \geq 0\}$ is a regular cosine family of continuous additive set-valued functions $A_t\colon K \to cc(K)$, $x \in A_t(x)$ for all $x \in K$, $t \geq 0$ and $A_t \circ A_s = A_s \circ A_t$ for all $s, t \geq 0$. Then this cosine family is twice differentiable and*

$$DA_t(x)|_{t=0} = \{0\}, \qquad D^2 A_t(x) = A_t(A(x))$$

*for $x \in K$, $t \geq 0$, where $DA_t(x)$ denotes the Hukuhara derivative of $A_t(x)$ with respect to $t$ and $A(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.*

We would like to obtain a similar result to the above one for a cosine family of continuous Jensen multifunctions. For this purpose we remind some properties of such a family.

Lemma 5 ([6], Theorem 3)
*Let $X$ be a Banach space and let $K$ be a closed convex cone in $X$ such that $\operatorname{int} K \neq \emptyset$. A one-parameter family $\{F_t : t \geq 0\}$ is a regular cosine family of continuous Jensen multifunctions $F_t\colon K \to cc(K)$ such that $x \in F_t(x)$ for all $x \in K$, $t \geq 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$ if and only if there exist a regular cosine family $\{A_t : t \geq 0\}$ of continuous additive multifunctions $A_t\colon K \to cc(K)$ such that $x \in A_t(x)$ for all $x \in K$, $t \geq 0$, $A_t \circ A_s = A_s \circ A_t$ for all $s, t \geq 0$ and a set $D \in cc(K)$ with zero for which conditions*

$$A_{t+s}(D) + A_{t-s}(D) = 2A_t(A_s(D)) \qquad \text{for } 0 \leq s \leq t,$$

$$F_t(x) = A_t(x) + \int_0^t \left( \int_0^s A_u(D)\, du \right) ds \qquad \text{for } t \geq 0$$

*hold.*

Using Lemmas 2, 3, 4 and 5 we obtain the following theorem.

Theorem 1
*Let $X$ be a Banach space and let $K$ be a closed convex cone with a nonempty interior in $X$. Suppose that $\{F_t : t \geq 0\}$ is a regular cosine family of continuous Jensen set-valued functions $F_t\colon K \to cc(K)$, $x \in F_t(x)$ for all $x \in K$, $t \geq 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$. Then this cosine family is twice differentiable and*

$$DF_t(x)|_{t=0} = \{0\}, \qquad D^2 F_t(x) = A_t(A(x) + D)$$

*for $x \in K$, $t \geq 0$, where $DF_t(x)$ denotes the Hukuhara derivative of $F_t(x)$ with respect to $t$, $D \in cc(K)$ with zero, $A(x) = D^2 A_t(x)|_{t=0}$, $\{A_t : t \geq 0\}$ is a regular cosine family of continuous additive multifunctions (as in Lemma 5).*

Let $K$ be a closed convex cone with a nonempty interior in $X$. We consider a continuous multifunction $\Phi\colon [0,+\infty)\times K \to cc(K)$ Jensen with respect to the second variable. According to Lemma 1 there exist multifunctions $A_\Phi\colon [0,+\infty)\times K \to cc(X)$ additive with respect to the second variable and $G_\Phi\colon [0,+\infty)\to cc(X)$ such that

$$\Phi(t,x) = A_\Phi(t,x) + G_\Phi(t) \qquad \text{for } x\in K,\ t\in[0,+\infty). \tag{2}$$

Setting $x=0$ in (2) we have

$$\Phi(t,0) = G_\Phi(t) \in cc(K) \qquad \text{for } t\in[0,+\infty).$$

Since $A_\Phi(t,x) + \frac{1}{n}G_\Phi(t) = \frac{1}{n}\Phi(t,nx) \subset K$ for all $n\in\mathbb{N}$ and the set $K$ is closed, $A_\Phi(t,x)\in cc(K)$ for $x\in K$, $t\in[0,+\infty)$. Moreover, multifunctions $A_\Phi, G_\Phi$ are continuous. Indeed, $t\mapsto G_\Phi(t)=\Phi(t,0)$ is continuous. As $\Phi$ and $G_\Phi$ are continuous, the multifunction $A_\Phi$ is also continuous.

Theorem 1 is a motivation for studying existence and uniqueness of a solution $\Phi\colon [0,+\infty)\times K\to cc(K)$, which is Jensen with respect to the second variable, of the following differential problem

$$\begin{aligned} \Phi(0,x) &= \Psi(x),\\ D\Phi(t,x)|_{t=0} &= \{0\},\\ D^2\Phi(t,x) &= A_\Phi(t,H(x)), \end{aligned} \tag{3}$$

where $H,\Psi\colon K\to cc(K)$ are given continuous Jensen set-valued functions, $D\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to $t$ and $A_\Phi$ is the additive, with respect to the second variable, part of $\Phi$.

Definition 1

A multifunction $\Phi\colon[0,+\infty)\times K\to cc(K)$ is said to be a solution of the problem (3) if it is continuous, twice differentiable with respect to $t$ and $\Phi$ satisfies (3) everywhere in $[0,+\infty)\times K$ and in $K$, respectively, where $H,\Psi\colon K\to cc(K)$ are two given continuous Jensen multifunctions.

With the problem (3), we associate the following equation

$$\Phi(t,x) = \Psi(x) + \int_0^t\left(\int_0^s A_\Phi(u,H(x))\,du\right)ds \tag{4}$$

for $x\in K$, $t\in[0,+\infty)$, where $H,\Psi\colon K\to cc(K)$ are given continuous Jensen multifunctions and $A_\Phi$ is the additive, with respect to the second variable, part of $\Phi$.

Definition 2

Let $H,\Psi\colon K\to cc(K)$ be two continuous Jensen set-valued functions. A map $\Phi\colon[0,+\infty)\times K\to cc(K)$ is said to be a solution of (4) if it is continuous and satisfies (4) everywhere.

Theorem 2

*Let $K$ be a closed convex cone with a nonempty interior in a Banach space and let $H, \Psi\colon K \to cc(K)$ be two continuous Jensen multifunctions. Let $\Phi\colon [0,+\infty)\times K \to cc(K)$ be a given Jensen with respect to the second variable set-valued function. This $\Phi$ is a solution of the problem (3) if and only if it is a solution of (4).*

The proof of Theorem 2 is the same as the proof of Theorem 1 in [5].

In the proof of the next theorem we use the following lemmas.

Lemma 6 ([12], Theorem 3)

*Let $X$ and $Y$ be two normed spaces and let $K$ be a convex cone in $X$. Suppose that $\{F_i : i \in I\}$ is a family of superadditive lower semicontinuous in $K$ and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i\colon K \to n(Y)$. If $K$ is of the second category in $K$ and $\bigcup_{i\in I} F_i(x) \in b(Y)$ for $x \in K$, then there exists a constant $M \in (0,+\infty)$ such that*

$$\sup_{i\in I} \|F_i(x)\| \le M\|x\| \qquad \text{for } x \in K.$$

Let $K$ be a closed convex cone in $X$. Applying Lemma 6 we can define the *norm* $\|F\|$ of a continuous additive multifunction $F\colon K \to n(K)$ to be the smallest element of the set

$$\{M > 0 : \ \|F(x)\| \le M\|x\|,\ x \in K\}.$$

Lemma 7

*Let $K$ be a closed convex cone with a nonempty interior in a Banach space and let $H, \Psi\colon K \to cc(K)$ be two continuous Jensen multifunctions. Assume that a continuous multifunction $A\colon [0,T]\times K \to cc(K)$ is additive with respect to the second variable. Then the multifunction*

$$F(t,x) := \Psi(x) + \int_0^t \left( \int_0^s A(u,H(x))\,du \right) ds, \qquad (t,x) \in [0,T]\times K \tag{5}$$

*is Jensen with respect to the second variable and continuous.*

*Proof.* The proof is based upon ideas found in the proof of Theorem 2 in the paper [5]. According to the proof of Theorem 1 in [5] we have that the multifunction $u \mapsto A(u,H(x))$ is continuous for all $x \in K$. We see that every set $F(t,x)$ belongs to $cc(K)$ and $F$ is Jensen with respect to the second variable.

Next we show that $F$ is continuous. Let $x, y \in K$ and $0 \le t_1 \le t_2 \le T$. The set

$$A([0,T],x) = \bigcup_{t\in[0,T]} A(t,x)$$

is compact (see [1], Ch. IV, p. 110, Theorem 3), so it is bounded. Therefore, by Lemma 6, there exists a positive constant $M_A$ such that

$$\|A(u,a)\| \le M_A\|a\| \tag{6}$$

for $u \in [0,T]$ and $a \in K$. This implies that

$$\|A(u,H(x))\| \leq M_A\|H(x)\|$$

for $u \in [0,T]$. Thus

$$\begin{aligned}
\left\| \int_{t_1}^{t_2} \left( \int_0^s A(u,H(x))\, du \right) ds \right\| &\leq \int_{t_1}^{t_2} \left( \int_0^s \|A(u,H(x))\|\, du \right) ds \\
&\leq \int_{t_1}^{t_2} \left( \int_0^s M_A\|H(x)\|\, du \right) ds \qquad (7) \\
&= \frac{t_2^2 - t_1^2}{2} M_A\|H(x)\|.
\end{aligned}$$

From Lemma 5 in [11] and (6) there exists a positive constant $M_0$ such that

$$h(A(u,a),A(u,b)) \leq M_0\|A(u,\cdot)\|\|a-b\| \leq M_0M_A\|a-b\|$$

for $u \in [0,T]$ and $a,b \in K$. Therefore,

$$A(u,a) \subset A(u,b) + M_0M_A\|a-b\|S$$

for $u \in [0,T]$ and $a,b \in K$.

Let $\varepsilon > 0$ and $a \in H(x)$. There exists $b \in H(y)$ for which

$$\|a-b\| < d(a,H(y)) + \frac{\varepsilon}{M_0M_A}.$$

This shows that for every $a \in H(x)$ there exists $b \in H(y)$ such that

$$\begin{aligned}
A(u,a) &\subset A(u,b) + M_0M_Ad(a,H(y))S + \varepsilon S \\
&\subset A(u,H(y)) + M_0M_Ah(H(x),H(y))S + \varepsilon S,
\end{aligned}$$

thus

$$A(u,H(x)) \subset A(u,H(y)) + M_0M_Ah(H(x),H(y))S + \varepsilon S$$

for $u \in [0,T]$. Since $\varepsilon > 0$ and $x,y \in K$ are arbitrary, we obtain

$$h(A(u,H(x)),A(u,H(y))) \leq M_0M_Ah(H(x),H(y)).$$

Hence and by properties of the Riemann integral we have

$$\begin{aligned}
&h\left( \int_0^t \left( \int_0^s A(u,H(x))\, du \right) ds, \int_0^t \left( \int_0^s A(u,H(y))\, du \right) ds \right) \\
&\leq \int_0^t \left( \int_0^s h(A(u,H(x)),A(u,H(y)))\, du \right) ds \qquad (8) \\
&\leq \int_0^t \left( \int_0^s M_0M_Ah(H(x),H(y))\, du \right) ds \\
&= \frac{t^2}{2} M_0M_Ah(H(x),H(y)).
\end{aligned}$$

By (5), (7) and (8) we get

$$\begin{aligned}
&h(F(t_1,x),F(t_2,y))\\
&\quad\le h(\Psi(x),\Psi(y))\\
&\qquad+h\left(\int_0^{t_1}\left(\int_0^s A(u,H(x))\,du\right)ds,\int_0^{t_2}\left(\int_0^s A(u,H(y))\,du\right)ds\right)\\
&\quad\le h(\Psi(x),\Psi(y))\\
&\qquad+h\left(\int_0^{t_1}\left(\int_0^s A(u,H(x))\,du\right)ds,\int_0^{t_1}\left(\int_0^s A(u,H(y))\,du\right)ds\right)\\
&\qquad+h\left(\{0\},\int_{t_1}^{t_2}\left(\int_0^s A(u,H(y))\,du\right)ds\right)\\
&\quad\le h(\Psi(x),\Psi(y))+\frac{t_1^2}{2}M_0M_Ah(H(x),H(y))+\frac{t_2^2-t_1^2}{2}M_A\|H(y)\|.
\end{aligned}$$

This shows that $F$ is a continuous set-valued function, because $\Psi$ and $H$ are continuous.

Theorem 3

*Let $K$ be a closed convex cone with a nonempty interior in a Banach space and let $H,\Psi\colon K\to cc(K)$ be two continuous Jensen multifunctions. Then there exists exactly one solution, Jensen with respect to the second variable, of the problem* (3).

*Proof.* Fix $T>0$. Let $E$ be the set of all continuous set-valued functions $\Phi\colon[0,T]\times K\to cc(K)$ such that $x\mapsto\Phi(t,x)$ are Jensen. As it was shown, for $\Phi\in E$ there exist continuous multifunctions $A_\Phi\colon[0,T]\times K\to cc(K)$ additive with respect to the second variable and $G_\Phi\colon[0,T]\to cc(K)$ such that $\Phi(t,x)=A_\Phi(t,x)+G_\Phi(t)$ for $x\in K$, $t\in[0,T]$.

Let $\Phi,\Pi\in E$ be given by

$$\Phi(t,x)=A_\Phi(t,x)+G_\Phi(t)\quad\text{and}\quad\Pi(t,x)=A_\Pi(t,x)+G_\Pi(t)\tag{9}$$

for $(t,x)\in[0,T]\times K$, where $A_\Phi,A_\Pi\colon[0,T]\times K\to cc(K)$ are additive with respect to the second variable and $G_\Phi(t),G_\Pi(t)\in cc(K)$. We define a functional $\rho$ in $E\times E$ as follows

$$\begin{aligned}\rho(\Phi,\Pi)=\sup\{h(A_\Phi(t,B),A_\Pi(t,B))+h(G_\Phi(t),G_\Pi(t)):\\ 0\le t\le T,\ B\in cc(K),\ \|B\|\le 1\}.\end{aligned}$$

We see that sets

$$A_i([0,T],x)=\bigcup_{t\in[0,T]}A_i(t,x),\qquad x\in K,$$

$$G_i([0,T])=\bigcup_{t\in[0,T]}G_i(t),$$

where $i \in \{\Phi, \Pi\}$ are compact (see [1], Ch. IV, p. 110, Theorem 3), so they are bounded. By Lemma 6 there exist positive constants $M_{A_\Phi}$ and $M_{A_\Pi}$ such that

$$\|A_\Phi(t,x)\| \le M_{A_\Phi}\|x\|, \qquad \|A_\Pi(t,x)\| \le M_{A_\Pi}\|x\|$$

for $t \in [0,T]$ and $x \in K$. We note that

$$\begin{aligned} &h(A_\Phi(t,B), A_\Pi(t,B)) + h(G_\Phi(t), G_\Pi(t)) \\ &\quad \le \|A_\Phi(t,B)\| + \|A_\Pi(t,B)\| + \|G_\Phi([0,T])\| + \|G_\Pi([0,T])\| \\ &\quad \le M_{A_\Phi} + M_{A_\Pi} + \|G_\Phi([0,T])\| + \|G_\Pi([0,T])\| \end{aligned}$$

for $t \in [0,T]$ and $B \in cc(K)$ such that $\|B\| \le 1$. Thus

$$\rho(\Phi, \Pi) < +\infty,$$

so the functional $\rho$ is finite. It is easy to verify that $\rho$ is a metric in $E$.

As the space $(cc(K), h)$ is a complete metric space (see [2]), $(E, \rho)$ is also a complete metric space.

We introduce the map $\Gamma$ which associates with every $\Phi \in E$ the set-valued function $\Gamma\Phi$ defined by

$$(\Gamma\Phi)(t,x) := \Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, H(x))\, du \right) ds$$

for $(t,x) \in [0,T] \times K$. We see that every set $(\Gamma\Phi)(t,x)$ belongs to $cc(K)$. By Lemma 7 the multifunction $\Gamma\Phi$ is Jensen with respect to the second variable and continuous. Therefore, $\Gamma\colon E \to E$.

Now, we prove that $\Gamma$ has exactly one fixed point. According to Lemma 1 we take the notations $\Psi(x) = A_\Psi(x) + G_\Psi$ and $H(x) = A_H(x) + G_H$, $x \in K$, where $A_\Psi, A_H\colon K \to cc(K)$ are additive and $G_\Psi, G_H \in cc(K)$. Let $\Phi, \Pi \in E$ be of the form (9) and let $(t,x) \in [0,T] \times K$. We observe that

$$\begin{aligned} (\Gamma\Phi)(t,x) &= \Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, H(x))\, du \right) ds \\ &= A_\Psi(x) + G_\Psi + \int_0^t \left( \int_0^s A_\Phi(u, A_H(x))\, du \right) ds \\ &\quad + \int_0^t \left( \int_0^s A_\Phi(u, G_H)\, du \right) ds, \end{aligned}$$

thus the additive part $A_{\Gamma\Phi}(t,x)$ of $\Gamma\Phi$ is equal to

$$A_\Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, A_H(x))\, du \right) ds$$

and similarly

$$A_{\Gamma\Pi}(t,x) = A_\Psi(x) + \int_0^t \left( \int_0^s A_\Pi(u, A_H(x))\, du \right) ds.$$

Hence and by properties of the Hausdorff metric we have

$$\begin{aligned} & h(A_{\Gamma\Phi}(t,x), A_{\Gamma\Pi}(t,x)) + h(G_{\Gamma\Phi}(t), G_{\Gamma\Pi}(t)) \\ & \quad = h\left( \int_0^t \left( \int_0^s A_\Phi(u, A_H(x))\, du \right) ds, \int_0^t \left( \int_0^s A_\Pi(u, A_H(x))\, du \right) ds \right) \\ & \qquad + h\left( \int_0^t \left( \int_0^s A_\Phi(u, G_H)\, du \right) ds, \int_0^t \left( \int_0^s A_\Pi(u, G_H)\, du \right) ds \right) \\ & \quad \le \frac{t^2}{2!} \rho(\Phi,\Pi) \|A_H(x)\| + \frac{t^2}{2!} \rho(\Phi,\Pi) \|G_H\| \\ & \quad \le 2\frac{t^2}{2!} \rho(\Phi,\Pi) \max\{\|A_H(x)\|, \|G_H\|\}. \end{aligned}$$

Suppose that

$$\begin{aligned} h(A_{\Gamma^n\Phi}(t,x), A_{\Gamma^n\Pi}(t,x)) + h(G_{\Gamma^n\Phi}(t), G_{\Gamma^n\Pi}(t)) \\ \le 2\frac{t^{2n}}{(2n)!} \rho(\Phi,\Pi) \max\{\|A_H(x)\|, \|G_H\|\}^n \end{aligned} \tag{10}$$

for some $n \in \mathbb{N}$. Then

$$\begin{aligned} & h(A_{\Gamma^{n+1}\Phi}(t,x), A_{\Gamma^{n+1}\Pi}(t,x)) + h(G_{\Gamma^{n+1}\Phi}(t), G_{\Gamma^{n+1}\Pi}(t)) \\ & \quad = h\left( \int_0^t \left( \int_0^s A_{\Gamma^n\Phi}(u, A_H(x))\, du \right) ds, \int_0^t \left( \int_0^s A_{\Gamma^n\Pi}(u, A_H(x))\, du \right) ds \right) \\ & \qquad + h\left( \int_0^t \left( \int_0^s A_{\Gamma^n\Phi}(u, G_H)\, du \right) ds, \int_0^t \left( \int_0^s A_{\Gamma^n\Pi}(u, G_H)\, du \right) ds \right) \\ & \quad \le \int_0^t \left( \int_0^s 2\frac{u^{2n}}{(2n)!} \rho(\Phi,\Pi) \max\{\|A_H(x)\|, \|G_H\|\}^{n+1}\, du \right) ds \\ & \quad = 2\frac{t^{2n+2}}{(2n+2)!} \rho(\Phi,\Pi) \max\{\|A_H(x)\|, \|G_H\|\}^{n+1}. \end{aligned}$$

This shows that (10) holds for all $n \in \mathbb{N}$. Therefore,

$$\rho(\Gamma^n\Phi, \Gamma^n\Pi) \le 2\frac{(T^2 \max\{\|A_H\|, \|G_H\|\})^n}{(2n)!} \rho(\Phi,\Pi), \qquad n \in \mathbb{N}.$$

We observe that for every $T > 0$ there exists $n \in \mathbb{N}$ such that

$$2\frac{(T^2 \max\{\|A_H\|, \|G_H\|\})^n}{(2n)!} < 1.$$

By Banach Fixed Point Theorem we get that $\Gamma^n$ has exactly one fixed point, whence it follows that $\Gamma$ has exactly one fixed point. This means that there exists exactly one solution of the problem (3) for $(t, x) \in [0, T] \times K$.

Now we give an application. Let $K$ be a closed convex cone with a nonempty interior in a Banach space. Suppose that $\{F_t : t \geq 0\}$ and $\{G_t : t \geq 0\}$ are regular cosine families of continuous Jensen multifunctions $F_t\colon K \to cc(K)$, $G_t\colon K \to cc(K)$ such that $x \in F_t(x)$, $x \in G_t(x)$, $F_t \circ F_s = F_s \circ F_t$, $G_t \circ G_s = G_s \circ G_t$ for $x \in K$, $s, t \geq 0$ and

$$H(x) := D^2 F_t(x)|_{t=0} = D^2 G_t(x)|_{t=0}.$$

Then multifunctions $(t, x) \mapsto F_t(x)$ and $(t, x) \mapsto G_t(x)$ are Jensen with respect to $x$ and satisfy (3) with $\Psi(x) = \{x\}$. According to Theorem 3 we have $F_t(x) = G_t(x)$ for $(t, x) \in [0, +\infty) \times K$. This means that if two regular cosine family as above have the same second order infinitesimal generator, then there are equal.

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# Regulated functions and integrability

**Abstract.** Properties of functions defined on a bounded closed interval, weaker than continuity, have been considered by many mathematicians. Functions having both sides limits at each point are called regulated and were considered by J. Dieudonné [2], D. Fraňková [3] and others (see for example S. Banach [1], S. Saks [8]). The main class of functions we deal with consists of piece-wise constant ones. These functions play a fundamental role in the integration theory which had been developed by Igor Kluvanek (see Š. Tkacik [9]). We present an outline of this theory.

## 1. Regulated functions

Everybody familiar with basic calculus remembers properties of continuous functions defined on a bounded closed interval. Some of those properties can be extended to suitably discontinuous functions, namely to functions having the right and the left limits at each point; such functions are called regulated. We shall deal with a special class of regulated functions consisting of piece-wise constant functions.

From now on, $I$ will denote a closed bounded interval $[a, b]$ of real numbers. All considered functions will be bounded and defined in the interval $I$.

A limit of a function is meant to be proper, i.e., different from $+\infty$ or $-\infty$.

Definition 1
A function $f\colon I \to \mathbb{R}$ is called regulated on $I$ if $f$ has the left-sided limit at every point of the interval $I$ except the point $a$ and $f$ has the right-sided limit at every point of the interval $I$ except the point $b$.

The idea of regulated functions can be spread out to functions defined in a subset of the interval $I$, namely to a set $E$, such that each point from the interval $I$ is left-sided and right-sided accumulation point of the set $E$. Nevertheless we are not concerned to such approach.

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In this definition we do not require that the right-sided limit and the left-sided limit of the function at a point are equal. The picture below shows an example of a regulated function on $I$.

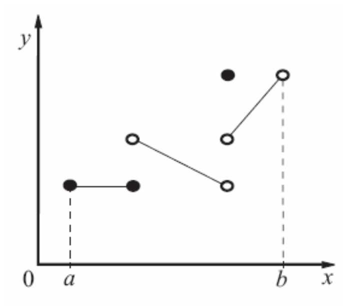


**Figure 1**

Important class of regulated functions consists of piece-wise constant ones.

Definition 2
A function $f: I \to \mathbb{R}$ is said to be a step function on $I$ whenever there exist: a positive integer $n$, a sequence of points $(c_1, \ldots, c_n)$ such that

$$a = c_0 < c_1 < \ldots < c_{j-1} < c_j < \ldots < c_n = b$$

and the function $f$ is constant on each interval $(c_{j-1}, c_j)$, $j = 1, 2, \ldots, n$.

An example of a step function is shown in Figure 2.

It follows from the definition that if $f$ is regulated on an interval $I$, then it is also regulated in each subinterval $J$ ($J \subseteq I$).

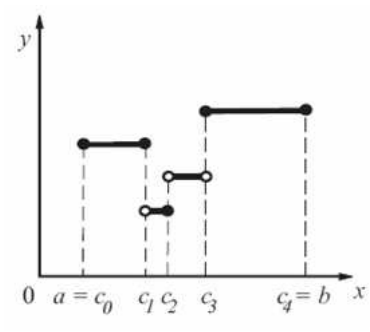


**Figure 2**

Although the next theorem is known (see [2] for example) we shall present an elementary proof of it.

This theorem states that a regulated function can be aproximated with arbitrary accuracy by a step function.

THEOREM 1
*Let $f\colon I \to \mathbb{R}$ be a regulated function and let $\varepsilon$ be a positive number. Then there exists a step function $g$ such that*

$$|f(x) - g(x)| < \varepsilon$$

*in each point $x$ of the interval $I$.*
*If the function $f$ is continuous on the interval $I$, then we can choose the function $g$ to be right-continuous at each point of the interval $[a, b)$ or to be left-continuous at each point of the interval $(a, b]$.*

*Proof.* Let $Z$ be the set of all numbers $z$ from the interval $I = [a, b]$ for which there exists a step function $g_z$ such that

$$|f(x) - g_z(x)| < \varepsilon \tag{1}$$

for every $x \in [a, z]$. If the function $f$ is continuous, then the step function $g_z$ is assumed to be right-continuous at every point of the interval $[a, z)$. Our aim is to show that $b \in Z$. If it is so we take $g_b$ for $g$.

We shall do that by showing that the supremum of the set $Z$ belongs to $Z$ and that it is equal to $b$. The set $Z$ has a supremum because it is not empty (the number $a$ surely belongs to it) and bounded from above (no element of the set $Z$ is greater than $b$). Then, let $s = \sup Z$.

1. We prove that $s \in Z$.

If $s = a$, then $s \in Z$. So now assume that $a < s$. Then the function $f$ has a left limit $k$ at $s$ and for a positive number $\varepsilon$ there exists a number $c < s$ such that

$$|f(x) - k| < \varepsilon \tag{2}$$

for every $x \in (c, s)$. Since $c < s$, there exists a number $z \in Z$ such that $c < z$.

Let $g_z$ be a step function such that (1) holds for every $x \in [a, z]$ and, if the function $f$ is continuous in $[a, b]$, let $g_z$ be left-continuous at every point of the interval $[a, z)$. Define the function $g_s$ by letting $g_s(s) = f(s)$, provided $s$ belongs to the domain of $f$, further $g_s(x) = k$ for every $x \in [z, s)$ and, finally, $g_s(x) = g_z(x)$ for every $x \in [a, z]$. Then $g_s$ is a step function such that (1) holds for every $x \in [a, s]$ and if the function $f$ happens to be continuous in $[a, b]$, then $g_s$ is right-continuous at every point of the interval $[a, s)$. Hence, $s \in Z$.

2. We prove that $s = b$.

Assume to the contrary that $s < b$.

Since $s < b$, the function $f$ has a right-sided limit $k$ at $s$ and there exists a number $d > s$ such that (2) holds for every $x \in (s, d)$. As $s \in Z$, there exists a step function $g_s$ such that (1) holds for every $x \in [a, s]$ and $g_s$ is left-continuous at every point of the interval $[a, s)$ in case when $f$ is continuous in $[a, b]$. Let $z$ be a number such that $s < z < d$. Let $g_z(x) = g_s(x)$ for every $x \in [a, s)$; let $g_z(s) = f(s)$; and let $g_z(x) = k$ for every $x \in (s, z)$. Then $g_z$ is a step function such that (1) holds for every $x \in [a, z]$ and $g_z$ is right-continuous at every point of the interval $[a, z)$ if the function $f$ is continuous in $[a, b]$. Hence $z \in Z$, which is a contradiction since $s < z$ and $s = \sup Z$.

Similar arguments can be used for the case, when we want the function $g$ to be left-continuous, simply apply the previous argument to the function $f(-x)$, when $x \in [-b, -a]$.

## 2. Examples

Example 1

During the first 19 weeks of the financial year, the wage of an employee was 186 Euro weekly. Than he was promoted and had 203,50 Euro weekly. A month before the end of the financial year, due to general salaries and wages increase, his wage was increased to 211,30 Euro weekly. This last month represents 4,4 working weeks (four full weeks and two working days, each representing 0,2 of a working week). Indicate how the weekly wage depends on time.

If we want to introduce a function indicating how the weekly wage of the employee depended on time we represent the year by the interval $[0, 52]$, taking a week for a unit of time. The function $f$ representing the dependence of the wage on time can then be defined in the following manner:

$$f(t) = \begin{cases} 186 & \text{for } t \in [0, 19], \\ 203,50 & \text{for } t \in \left(19, 47\frac{3}{5}\right), \\ 211,30 & \text{for } t \in \left[47\frac{3}{5}, 52\right]. \end{cases}$$

If $\chi_A(t)$ is a characteristic function of the set $A$, then we have

$$f(t) = 186 \cdot \chi_{[0,19]}(t) + 203,50 \cdot \chi_{(19,47\frac{3}{5})}(t) + 211,30 \cdot \chi_{[47\frac{3}{5},52]}(t)$$

for every $t \in [0, 52]$.

Now we can ask what was the average (mean) wage of that employee during the year or what was his total income from wages in that year? Clearly, his total income was

$$186 \cdot 19 + 203,50 \cdot (47,6 - 19) + 211,30 \cdot (52 - 47,6) = 10283,82$$

Euro. His average wage was

$$\frac{10283,82}{52} = 197,76$$

Euro per week (rounded to whole cents). In this example it is easy to see that the function $f$ is a step function and it does not matter, if we use open or bounded intervals for calculating of the total income.

Here we defined $c_1 = 186$; $c_2 = 203,50$; $c_3 = 211,30$; $J_1 = [0, 19]$, $J_2 = [19, 47\frac{3}{5}]$, $J_3 = [47\frac{3}{5}, 52]$. If the number $b - a = \lambda(J)$ is the length of the interval $J = [a, b]$, then the total income has the form

$$c_1\lambda(J_1) + c_2\lambda(J_2) + c_3\lambda(J_3) = \sum_{j=1}^{3} c_j\lambda(J_j).$$

This number is also the area of the set $S = \{(t, y) : \ t \in [0, 52], 0 \le y \le f(t)\}$.

Therefore, it is possible to express the step function by the formula

$$f(x) = \sum_{j=1}^{n} c_j \chi_{J_j}(x)$$

for every $x$ in an interval $I$, where $n$ is a positive integer, $c_j$ are arbitrary numbers and $J_j$ some bounded intervals ($\bigcup_{j=1}^{n} J_j = I$) for every $j = 1, 2, 3, \ldots, n$. In each case, the number

$$\sum_{j=1}^{n} c_j \lambda(J_j)$$

is called the integral of the function $f$.

Example 2
Now, we try to calculate the area of the set

$$S = \{(x, y) : \ x \in I, \, 0 \le y \le f(x)\},$$

where $f$ is some continuous and non-negative function in the (compact) interval $I$.

If the function $f$ is not a step function in the interval $I$, then the set $S$ is not equal to the union of finite number of rectangles. Nevertheless, with the exception of some points on the boundary, which may be disregarded when calculating the area, this set can be covered by an infinite sequence of non-overlapping rectangles as illustrated in Figure 3. The sum of the areas of these rectangles is equal to the area of S.

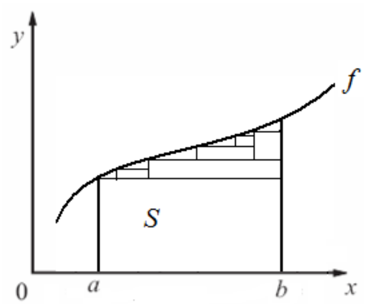


**Figure 3**

That is, there exist intervals $J_j \subset I$ and numbers $c_j$, $j = 1, 2, 3, \ldots$, such that

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x) \tag{3}$$

for every $x \in I$ and the area of set $S$ is equal to the number

$$\sum_{j=1}^{\infty} c_j \lambda(J_j). \tag{4}$$

The class of functions to which the procedure can be applied is much larger than in the case when $c_j \ge 0$ for every $j = 1, 2, 3, \ldots$. In particular, we now may consider functions with both positive and negative values. Consequently, we can also calculate the integral (4) of a function $f$ when it has an interpretation different from that of the area of a planar figure. Of course, if so desired, the integral of a function in an interval $I$ can always be interpreted "geometrically" as a difference of the areas of the sets

$$S^+ = \{(x, y) :\ x \in I,\, 0 \le y \le f(x)\} \quad \text{and} \quad S^- = \{(x, y) :\ x \in I,\, f(x) \le y \le 0\}.$$

## 3. Definition of the integral

To obtain a workable definition of integral for a sufficiently large class of functions, it suffices to require the existence of the sum (4) and to note that this sum is then independent of the particular choice of the numbers $c_j$ and intervals $J_j$, $j = 1, 2, 3, \dots$, used in the representation (3) of the function $f$.

Definition 3
A function $f$ is said to be integrable in the interval $I$ whenever there exist numbers $c_j$ and bounded intervals $J_j \subset I$, $j = 1, 2, 3, \dots$, such that

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty \tag{5}$$

and the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for every $x \in I$ such that

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty. \tag{6}$$

Now we shall introduce the notions of a virtually primitive function. We shall use the term *a condition $\mathcal{P}$ is fulfilled nearly everywhere*. It means that the set of points for which the condition $\mathcal{P}$ is not fulfilled is at most countable.

Definition 4
A function $F$ is said to be virtually primitive to a function $f$ in an interval $I$, if the function $F$ is continuous in the interval $I$ and $F'(x) = f(x)$ nearly everywhere in $I$.

In this definition we do not require $I$ to be a compact interval, it can be as well an unbounded interval.

We shall prove that if a function $f$ is integrable in the interval $I$, then the sum (4) is the same for every choice of the numbers $c_j$ and intervals $J_j$, $j = 1, 2, 3, \dots$, satisfying the condition (5), such that (3) holds for every $x \in I$ for which the inequality (6) does hold.

The next three theorems, which are technical ones, are useful in the proof that the definition of the Kluvanek integral is correct.

Theorem 2
*Let $n$ be a positive integer, $c_j$ non-negative numbers, $J_j$ bounded subintervals of $I$, $j = 1, 2, 3, \ldots, n$, $d_k$ non-negative numbers and $K_k$ bounded intervals, $k = 1, 2, 3, \ldots$, such that*

$$\sum_{j=1}^{n} c_j \chi_{J_j}(x) \le \sum_{k=1}^{\infty} d_k \chi_{K_k}(x) \tag{7}$$

*for every $x \in (-\infty, \infty)$. Then*

$$\sum_{j=1}^{n} c_j \lambda(J_j) \le \sum_{k=1}^{\infty} d_k \lambda(K_k). \tag{8}$$

*Proof.* It follows from the assumptions that $a$ is a number not greater than the left end-point and $b$ is a number not less than the right end-point of each of the intervals $J_j$, $j = 1, 2, 3, \ldots, n$. Let $F_j$ be a function virtually primitive in $(-\infty, \infty)$ to the function $c_j \chi_{J_j}$ such that $F_j(a) = 0$, $j = 1, 2, 3, \ldots, n$, and $G_k$ the function virtually primitive to $d_k \chi_{K_k}$ such that $G_k(a) = 0$, $k = 1, 2, 3, \ldots$. Since $c_j \lambda(J_j) = F_j(b)$, $j = 1, 2, 3, \ldots, n$, if we prove that

$$\sum_{j=1}^{n} F_j(b) \le \sum_{k=1}^{\infty} G_k(b),$$

then (8) will follow.

Suppose to the contrary that

$$\sum_{k=1}^{\infty} G_k(b) < \sum_{j=1}^{n} F_j(b). \tag{9}$$

First note that $0 \le G_k(x) \le G_k(b)$ for every $x \in [a, b]$ and every $k = 1, 2, 3, \ldots$. Hence, by (9), the sequence of functions $\{G_k\}_{n=1}^{\infty}$ is uniformly convergent in the interval $[a, b]$. Let

$$F(x) = \sum_{j=1}^{n} F_j(x) \quad \text{and} \quad G(x) = \sum_{k=1}^{\infty} G_k(x)$$

for every $x \in [a, b]$. The functions $F_j(x)$, $j = 1, 2, 3, \ldots, n$, and $G_k(x)$, $k = 1, 2, 3, \ldots$, are continuous in the interval $[a, b]$. Therefore, the functions $F(x)$ and $G(x)$ are also continuous in the interval $[a, b]$ and, of course, $F(a) = G(a) = 0$. Let

$$k = \frac{F(b) - G(b)}{2(b - a)} \quad \text{and} \quad q = \frac{F(b) - G(b)}{2}.$$

By (9), $k > 0$ and $q > 0$. If $t \in (0, k)$, let

$$h_t(x) = F(x) - G(x) - t(x - a) - q$$

for every $x \in [a, b]$. Then, for every $t \in (0, k)$, $h_t$ is a function continuous in the interval $[a, b]$ such that $h_t(a) < 0$ and $h_t(b) > 0$. Let $\xi(t)$ be its maximal root in the interval $(a, b)$. That is $h_t(\xi(t)) = 0$ and $h_t(y) > 0$ for every $y \in (\xi(t), b)$.

The function $\xi(t), t \in (0,k)$, is (strictly) increasing, because if $0 < t < s < k$, then

$$h_s(\xi(t)) = h_s(\xi(t)) - h_t(\xi(t)) = (t-s)(\xi(t)-a) < 0$$

and, hence, the largest root, $\xi(s)$, of the function $h_s$ is greater than $\xi(t)$. So, this function is injective. Since its domain, $(0,k)$, is not a countable set, the set of its values $\{\xi(t) : t \in (0,k)\}$ is not countable either. But the set of end-points of all intervals $J_j$, $j = 1,2,3,\ldots,n$, and $K_k, k = 1,2,3,\ldots$, is countable. So, there is a number $t \in (0,k)$ such that $\xi(t)$ is not an end-point of any of intervals $J_j$, $j = 1,2,3,\ldots,n$, and $K_k, k = 1,2,3,\ldots$. Let $t$ be such a number and $x = \xi(t)$, the corresponding point of the interval $(a,b)$. Then $h_t(x) = 0$ and $h_t(y) > 0$ for every $y \in (x,b)$. That is,

$$F(x) - G(x) = t(x-a) - q \quad \text{and} \quad F(y) - G(y) > t(y-a) - q$$

for every $y \in (x,b)$. Consequently,

$$\frac{F(y)-F(x)}{y-x} - \frac{G(y)-G(x)}{y-x} > t \tag{10}$$

for every $y \in (x,b]$.

On the other hand, since $x$ is not an end-point of any of the intervals $J_j$ and $K_k$, each function $F_j$ and $G_k$ is differentiable at $x$ and $F_j'(x) = c_j\chi_{J_j}(x)$ for $j = 1,2,3,\ldots,n$ and $G_k'(x) = d_k\chi_{K_k}(x)$ for $k = 1,2,3,\ldots$. So, by (7),

$$F'(x) = \sum_{j=1}^{n} F_j'(x) \leq \sum_{k=1}^{\infty} G_k'(x).$$

Since $t > 0$, there exists a positive integer $m$ such that

$$F'(x) \leq \sum_{k=1}^{\infty} G_k'(x) < \sum_{k=1}^{m} G_k'(x) + t.$$

Therefore,

$$\lim_{y\to x^+}\left(\frac{F(y)-F(x)}{y-x} - \sum_{k=1}^{m}\frac{G_k(y)-G_k(x)}{y-x}\right) < t.$$

From the properties of limits we have, that there exists a point $y$ in the interval $[x,b]$ such that

$$\frac{F(y)-F(x)}{y-x} - \sum_{k=1}^{m}\frac{G_k(y)-G_k(x)}{y-x} < t. \tag{11}$$

Now, $G_k(y) - G_k(x) > 0$ for every $k = m+1, m+2, \ldots$, because the functions $G_k$ are non-decreasing. Hence,

$$\frac{G(y)-G(x)}{y-x} = \sum_{k=1}^{\infty}\frac{G_k(y)-G_k(x)}{y-x} \geq \sum_{k=1}^{m}\frac{G_k(y)-G_k(x)}{y-x}.$$

So, (11) contradicts (10).

THEOREM 3
*Let $c_j$ and $d_j$ be non-negative numbers and let $J_j$ and $K_j$ be subintervals of $I$, $j = 1, 2, 3, \dots$, such that*

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) < \infty, \qquad \sum_{j=1}^{\infty} d_j \lambda(K_j) < \infty$$

*and*

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) \tag{12}$$

*for every $x$ for which*

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) < \infty.$$

*Then*

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j). \tag{13}$$

*Proof.* Let $\varepsilon$ be an arbitrary positive number. Let $n$ be a positive integer such that

$$\sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \frac{\varepsilon}{2}.$$

Then

$$\sum_{j=1}^{n} c_j \chi_{J_j}(x) \le \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) + \sum_{j=n+1}^{\infty} c_j \chi_{J_j}(x)$$

for every $x \in (-\infty, \infty)$ with no exception.

By Theorem 2,

$$\sum_{j=1}^{n} c_j \lambda(J_j) \le \sum_{j=1}^{\infty} d_j \lambda(K_j) + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \sum_{j=1}^{\infty} d_j \lambda(K_j) + \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{\infty} c_j \lambda(J_j) &= \sum_{j=1}^{n} c_j \lambda(J_j) + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) \\
&< \sum_{j=1}^{\infty} d_j \lambda(K_j) + \frac{\varepsilon}{2} + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) \\
&< \sum_{j=1}^{\infty} d_j \lambda(K_j) + \varepsilon.
\end{aligned}$$

Because the inequality between the first and the last term holds for every positive $\varepsilon$, we have

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) \leq \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

The reverse inequality can be proved by a symmetric argument. Hence (13) holds.

Recall that nonnegative $x^+$ and nonpositive $x^-$ parts of a number $x$ are defined by

$$x^+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad x^- = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then: $x^+ \geq 0$, $x^- \geq 0$, $x = x^+ - x^-$ and $|x| = x^+ + x^-$ for any real number $x$.

Theorem 4

*Let $c_j$ and $d_j$ be real numbers and let $J_j$ and $K_j$, $j = 1, 2, \ldots$, be subintervals of $I$ such that*

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty, \qquad \sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty. \tag{14}$$

*If*

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x)$$

*for every $x \in I$ for which*

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty,$$

*then*

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

*Proof.* The conditions (14) imply:

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} c_j^- \lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) < \infty, \quad \sum_{j=1}^{\infty} d_j^- \lambda(J_j) < \infty.$$

From condition

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x)$$

we have

$$\sum_{j=1}^{\infty} c_j^+ \chi_{J_j}(x) - \sum_{j=1}^{\infty} c_j^- \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j^+ \chi_{K_j}(x) - \sum_{j=1}^{\infty} d_j^- \chi_{K_j}(x).$$

That is

$$\sum_{j=1}^{\infty} c_j^+ \chi_{J_j}(x) + \sum_{j=1}^{\infty} d_j^- \chi_{K_j}(x) = \sum_{j=1}^{\infty} d_j^+ \chi_{K_j}(x) + \sum_{j=1}^{\infty} c_j^- \chi_{J_j}(x)$$

for every $x$ such that both sides represent a real number (not $\infty$). By Theorem 3

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) + \sum_{j=1}^{\infty} d_j^- \lambda(K_j) = \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) + \sum_{j=1}^{\infty} c_j^- \lambda(J_j),$$

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) - \sum_{j=1}^{\infty} c_j^- \lambda(J_j) = \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) - \sum_{j=1}^{\infty} d_j^- \lambda(K_j)$$

and

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

Now we are able to proceed with the definition of integral:

Definition 5

Let $f$ be a function integrable in the interval $I$. Let $c_j$ be numbers and let $J_j \subset I$ be intervals, $j = 1, 2, 3, \ldots$, satisfying the condition

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

such that the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for every $x \in I$ satisfying the condition

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty.$$

Then the number

$$\sum_{j=1}^{\infty} c_j \lambda(J_j)$$

is called the integral of $f$ in the interval $I$; it will be denoted by $\int_I f(x)\, dx$.

Clearly, for every constant function $f(x) = \beta$ in the interval $[a, b]$ we have

$$\int_a^b f(x)\, dx = \beta(b - a).$$

## 4. Integration of regulated functions

The next theorem shows how to integrate regulated functions.

THEOREM 5
*Let $f$ be a regulated function in the interval $[a,b]$ $(a<b)$. Then $f$ is integrable in this interval and*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

*where $F$ is any virtually primitive function to $f$ in the interval $[a,b]$.*

*Proof.* Let $\{f_n(x)\}_{n=1}^{\infty}$ be a uniformly convergent sequence of step functions in the interval $[a,b]$ such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for every $x \in [a,b]$. This sequence exists from the theory of regulated and piecewise constant functions (see [5]). The functions $f_n(x)$ are bounded. Let

$$\beta_n = \sup\{|f_n(x)| : \ x \in I\}$$

for every $n = 1, 2, 3, \ldots$.

It follows from the uniform convergence of the sequence $\{f_n(x)\}_{n=1}^{\infty}$ that

$$\sum_{n=1}^{\infty} \beta_n < \infty. \tag{15}$$

For every $n = 1, 2, 3, \ldots$ we have

$$\int_a^b |f_n(x)|\,dx \le \int_a^b \beta_n\,dx = \beta_n(b-a).$$

From (15) we have $\sum_{n=1}^{\infty} \int_a^b |f_n(x)|\,dx < \infty$. The function $f$ is integrable in the interval $[a,b]$ and

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx. \tag{16}$$

Let $F_n$ be a function virtually primitive to the function $f_n$ in the interval $[a,b]$ such that $F_n(a) = 0$ for $n = 1, 2, 3, \ldots$. The sum

$$F(x) = \sum_{n=1}^{\infty} F_n(x)$$

exists for every $x \in [a,b]$ and function $F$ defined in this way is virtually primitive to $f$ in $[a,b]$. Thus

$$\int_a^b f_n(x)\,dx = F_n(b) - F_n(a)$$

holds for every $n = 1, 2, 3, \ldots$. Hence by (16)

$$\int_a^b f_n(x)\,dx = \sum_{n=1}^{\infty} (F_n(b) - F_n(a)) = F(b) - F(a).$$

Since the difference of any two functions virtually primitive to $f$ in $[a,b]$ is constant, the last equality holds for any function $F$ virtually primitive to $f$ in $[a,b]$.

## 5. Conclusions

Our aim was to provide an introduction to the theory of integral developed by Professor Igor Kluvanek during his stay at Flinders University in Adelaide (Australia). In his approach regulated functions play an important role (see I. Kluvánek [4]).

The definition of integral given in this article applies an idea of Archimedes. The most effective method for the calculation of integrals is the one which is based on differential calculus (see V.V. Mityushev, S.V. Rogosin [6] and W.F. Pfeffer [7]).

As everybody knows Dirichlet function (characteristic function of the set of rational numbers) is not integrable in Riemann sense. It is possible to show, that this function is integrable in the sense of I. Kluvanek and the value of this integral is zero. In fact, let $\mathbb{Q} \cap [a,b] = \{q_j : j \in \mathbb{N}\}$. Let further $J_{2j} = \{q_j\}$ and let $J_{2j-1}$ be any subintervals of $[0,1]$. Hence the Dirichlet function $D\colon [0,1] \to \mathbb{R}$ can be represented in the form

$$D(x) = \sum_{j=1}^{\infty} c_j \cdot \chi_{J_j}(x),$$

where $c_{2j} = 1$ and $c_{2j-1} = 0$. Hence its integral equals 0.

Applying properties of this kind of integral it is possible to prove that integral of a regulated function $f$ is an additive function of interval.

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# A note on some iterative roots

**Abstract.** In this paper some orientation-preserving iterative roots of an orientation-preserving homeomorphism $F\colon S^1 \to S^1$ which possess periodic points of order $n$ are considered. Namely, iterative roots with periodic points of order $n$. All orders of such roots are determined and their general construction is given.

Let $X$ be a nonempty set. A function $g\colon X \to X$ is called an *iterative root* of a given function $f\colon X \to X$ if $g^m(x) = f(x)$ for $x \in X$. The number $m \geq 2$ is called the *order of the iterative root* and $g^m$ denotes $m$-th iterate of $g$. Moreover, we say that $x \in X$ is a *periodic point* of $f$ of order $n \in \mathbb{N}$, $n > 1$ if

$$f^n(x) = x \quad \text{and} \quad f^k(x) \neq x \text{ for } k \in \{1, \dots, n-1\}.$$

If $f(x) = x$, then $x$ is said to be a *fixed point of* $f$. The set of all periodic (fixed) points of $f$ will be denoted by $\operatorname{Per} f$ $(\operatorname{Fix} f)$.

In [9] M.C. Zdun proved that every orientation-preserving homeomorphism $F\colon S^1 \to S^1$ possessing periodic points of order $n$ is a composition of two orientation-preserving homeomorphisms $T, G\colon S^1 \to S^1$. Function $G$ has no periodic points except fixed points and $T$ is such that $T^n = \mathrm{id}_{S^1}$. Using this result he determined all continuous iterative roots with periodic points for homeomorphisms having fixed points.

In the present paper we apply Zdun's theorem to the problem of finding some continuous iterative roots for an orientation-preserving homeomorphism $F\colon S^1 \to S^1$ with periodic points of order $n$. Namely, we shall give conditions under which continuous iterative roots with periodic points of order $n$ exist and give the construction of these roots.

Now, we recall some useful notations and definitions related to the mappings of the circle. Let $u, w \in S^1$ and $u \neq w$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2 < t_1 + 1$ and $u = e^{2\pi i t_1}$ and $w = e^{2\pi i t_2}$. Put

$$\overrightarrow{(u,w)} := \left\{e^{2\pi i t} :\ t \in (t_1, t_2)\right\}, \quad \overrightarrow{[u,w]} := \overrightarrow{(u,w)} \cup \{u, w\}, \quad \overrightarrow{[u,w)} := \overrightarrow{(u,w)} \cup \{u\}.$$

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These sets are called arcs.

For every homeomorphism $F\colon S^1 \to S^1$ there exists a unique (up to translation by an integer) homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$F\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} f(x)}$$

and

$$f(x+1) = f(x) + k$$

for all $x \in \mathbb{R}$, where $k \in \{-1, 1\}$. We call $F$ *orientation-preserving* if $k = 1$, which is equivalent to the fact that $f$ is increasing.

Moreover, for every continuous function $G\colon I \to J$, where $I = \{e^{2\pi \mathrm{i} t} : t \in [a,b]\}$ and $J = \{e^{2\pi \mathrm{i} t} : t \in [c,d]\}$ there exists a unique continuous function $g\colon [a,b] \to [c,d]$ such that

$$G\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} g(x)}, \qquad x \in [a,b].$$

In this case we also call $g$ the lift of $G$ and we say that $G$ preserves orientation if $g$ is strictly increasing.

For any orientation-preserving homeomorphism $F\colon S^1 \to S^1$, the limit

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1, \qquad x \in \mathbb{R}$$

always exists and does not depend on the choice of $x$ and $f$. This number is called the *rotation number* of $F$ (see [3]). It is known that $\alpha(F)$ is a rational and positive number if and only if $F$ has a periodic point (see for example [3]). If $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where $q, n$ are positive integers with $q < n$ and $\gcd(q,n) = 1$, then $\operatorname{Per} F$ contains only periodic points of order $n$ (see [7], [5]). Moreover, there exists a unique number $p \in \{1, \ldots, n-1\}$, called the *characteristic number of* $F$, satisfying $pq = 1 \pmod n$. From now on put $n_F := n$ and $\operatorname{char} F := p$. The following result comes from [8].

Lemma 1
*If $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism with $\operatorname{Per} F \neq \emptyset$, then for every $z \in \operatorname{Per} F$,*

$$\operatorname{Arg} \frac{F^{k \operatorname{char} F}(z)}{z} < \operatorname{Arg} \frac{F^{(k+1) \operatorname{char} F}(z)}{z}, \qquad k = 0, \ldots, n_F - 2.$$

For fixed $z \in \operatorname{Per} F$ we define the partition of $S^1$ onto the following arcs

$$I_k = I_k(z) := \overrightarrow{\left[F^{k \operatorname{char} F}(z), F^{(k+1)\operatorname{char} F}(z)\right)}, \qquad k \in \{0, \ldots, n_F - 1\}. \tag{1}$$

Let us note that

$$\begin{aligned} F[I_k] &= \overrightarrow{\left[F^{k \operatorname{char} F + 1}(z), F^{(k+1)\operatorname{char} F + 1}(z)\right)} \\ &= \overrightarrow{\left[F^{k \operatorname{char} F + q \operatorname{char} F}(z), F^{(k+1)\operatorname{char} F + q \operatorname{char} F}(z)\right)} \\ &= I_{(k+q) \pmod{n_F}}, \qquad\qquad k \in \{0, \ldots, n_F - 1\}, \end{aligned}$$

where $q = n_F \alpha(F)$.

We shall use the following property (see [9]).

Remark 1
Let $n \in \mathbb{N}$ and $p, q \in \{0, \dots, n-1\}$ satisfy $pq = 1 \,(\mathrm{mod}\, n)$ and $\gcd(q, n) = 1$. The mapping $\{0, \dots, n-1\} \ni d \mapsto i_d := -dp \,(\mathrm{mod}\, n) \in \{0, \dots, n-1\}$ is a bijection. Moreover, $d + i_d q = 0 \,(\mathrm{mod}\, n)$.

The next theorem also comes from [9] and it is a modification of the factorization theorem (see [9], Theorems 5 and 9).

Theorem 1
*Let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism, $z \in \operatorname{Per} F$ and let $\{I_d\}_{d\in\{0,\dots,n_F-1\}}$ be the family defined in (1). Then there exists a unique orientation-preserving homeomorphism $T\colon S^1 \to S^1$ having periodic points of order $n_F$ and such that $\operatorname{Per} T = S^1$ and*

$$F^{k+jn_F}_{|I_d} = T^{\alpha(F)n_F k} \circ \begin{cases} T^d \circ (F^{n_F})^{j+1} \circ T^{-d}_{|I_d}, & \text{if } i_d \le k-1, \\ T^d \circ (F^{n_F})^{j} \circ T^{-d}_{|I_d}, & \text{if } i_d > k-1 \end{cases}$$

*for $d, k \in \{0, 1, \dots, n_F - 1\}$, $j \in \mathbb{N}$.*

Let us stress that $T$ is unique up to a periodic point of $F$. Moreover, $F^{n_F}[I_d] = I_d$, $T[I_d] = I_{(d+1)\,(\mathrm{mod}\, n)}$ for $d \in \{0, \dots, n_F - 1\}$ and $T^{n_F} = \mathrm{id}_{S^1}$. Such a function $T$ will be called a *Babbage function of* $F$ (see [9]).

In view of the above theorem (see also [9], Corollary 6) for every orientation-preserving homeomorphism $F\colon S^1 \to S^1$ with $\emptyset \ne \operatorname{Per} F$ and for every $z_0 \in \operatorname{Per} F$ we have

$$F(z) := \begin{cases} T^q\left(F^{n_F}(z)\right), & z \in I_0(z_0), \\ T^q(z), & z \in S^1 \setminus I_0(z_0), \end{cases} \tag{2}$$

where $q = \alpha(F)n_F$ and $T$ is a Babbage function of $F$.

We start with the following

Remark 2
Let $n, m \ge 2$ be integers and let $q, q' \in \{1, \dots, n-1\}$ be such that $\gcd(q, n) = 1$ and $mq' = q \,(\mathrm{mod}\, n)$, then $\gcd(m, n) = 1$.

*Proof.* To obtain a contradiction suppose that $m = ka$ and $n = kb$ for some integers $k > 1$ and $a, b \ge 1$. This and the fact that $mq' = q \,(\mathrm{mod}\, n)$ give $kaq' = q + jkb$ for some $j \in \mathbb{Z}$. Therefore $k(aq' - jb) = q$, which contradicts the fact that $\gcd(q, n) = 1$.

Remark 3
Let $n, m \ge 2$ be relatively prime integers and let $q \in \{1, \dots, n-1\}$ be such that $\gcd(q, n) = 1$. There is a unique $q' \in \{1, \dots, n-1\}$ such that $\gcd(q', n) = 1$ and $mq' = q \,(\mathrm{mod}\, n)$.

*Proof.* The fact that $\gcd(m, n) = 1$ implies that the equation $mx + ny = q$ has integer solutions $x, y$. In particular, there is exactly one pair $(q', j)$, where $q' \in \{0, \dots, n-1\}$ and $j \in \mathbb{Z}$ such that $mq' + jn = q$. Thus $mq' = q \,(\mathrm{mod}\, n)$. Moreover, $q' \ne 0$ as $\gcd(q, n) = 1$. In the same manner as in the proof of Remark 2 we can see that $\gcd(q', n) = 1$.

From Remark 2 we can conclude that

Corollary 1
*Let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with $\emptyset \neq \operatorname{Per} F$ and let $m \geq 2$ be an integer. If equation*

$$G^m(z) = F(z), \qquad z \in S^1 \tag{3}$$

*has continuous and orientation-preserving solution such that $n_G = n_F$, then*

$$\gcd(m, n_F) = 1.$$

It appears that $\gcd(m, n_F) = 1$ is also a sufficient condition for the existence of continuous and orientation-preserving solutions of (3) with $n_G = n_F$. The proof of this property and the description of the solution of (3) in the case $\operatorname{Per} F = S^1$ can be found in [6]. Therefore, from now on assume that $\operatorname{Per} F \neq S^1$. Before we present some results let as recall that if (3) holds, then $\operatorname{Per} F = \operatorname{Per} G$.

Lemma 2
*Let $F, G\colon S^1 \to S^1$ be orientation-preserving homeomorphisms possessing periodic points of order $n_F = n_G = n$ and satisfying equation (3) for an $m \geq 2$. Let moreover $z_0 \in \operatorname{Per} F = \operatorname{Per} G$ and $J_k := \overrightarrow{[G^{k \operatorname{char} G}(z_0), G^{(k+1) \operatorname{char} G}(z_0))}$, $k \in \{0, \dots, n-1\}$. Then*

(i) *$J_k = I_k(z_0)$ for $k \in \{0, \dots, n-1\}$, where the arcs $I_k(z_0)$ are defined by (1);*

(ii) *$(G^{n_G})^m = F^{n_F}$;*

(iii) *if $T$ and $V$ are Babbage functions of $F$ and $G$, respectively, $[x]$ stands for an integer part of $x \in \mathbb{R}$ and $i'_d := -d \operatorname{char} G \pmod n$ for $d \in \{0, \dots, n-1\}$, then*

$$V^q_{|I_d} = T^{\alpha(F)n} \circ T^d \circ G^{n\beta_d} \circ T^{-d}_{|I_d}, \tag{4}$$

*where*

$$\beta_d := \begin{cases} m - \left[\frac{m}{n}\right] - 1, & d = 0, \\ -\left[\frac{m}{n}\right] - 1, & d \in \{1, \dots, n-1\},\ i'_d \leq m - \left[\frac{m}{n}\right] n - 1, \\ -\left[\frac{m}{n}\right], & d \in \{1, \dots, n-1\},\ i'_d > m - \left[\frac{m}{n}\right] n - 1. \end{cases} \tag{5}$$

*Proof.* Fix $z_0 \in \operatorname{Per} F$ and assume that (3) holds, $n_F = n_G = n$. Put $q := \alpha(F)n$, $q' := \alpha(G)n$ and $b := \left[\frac{m}{n}\right]$. From the fact that $\gcd(m, n) = 1$ (see Corollary 1), we get

$$m = k + bn \qquad \text{for some } k \in \{1, \dots, n-1\}. \tag{6}$$

To prove (i) it suffices to show that $G^{\operatorname{char} G}(z_0) = F^{\operatorname{char} F}(z_0)$. Equation (3) yields $m\alpha(G) = \alpha(F) \pmod 1$ (see [2]). Thus $mq' = q \pmod n$, hence

$$mq' \operatorname{char} F \operatorname{char} G = q \operatorname{char} F \operatorname{char} G \pmod n$$

and finally, in view of the definition of $\operatorname{char} F$,

$$m \operatorname{char} F = \operatorname{char} G \pmod{n}. \tag{7}$$

From (7), (3) and since $z_0$ is a periodic point of $G$ of order $n$ we obtain

$$G^{\operatorname{char} G}(z_0) = G^{m \operatorname{char} F}(z_0) = F^{\operatorname{char} F}(z_0).$$

Note that (ii) is an immediate consequence of equation (3) and equality $n_F = n_G$.

Now we prove (iii). From Theorem 1, (6) and (i) we get

$$G^m_{|I_d} = G^{k+bn}_{|I_d} = V^{q'k} \circ \begin{cases} V^d \circ (G^n)^{b+1} \circ V^{-d}_{|I_d}, & \text{if } i'_d \le k-1, \\ V^d \circ (G^n)^b \circ V^{-d}_{|I_d}, & \text{if } i'_d > k-1 \end{cases}$$

for $d \in \{0, 1, \ldots, n-1\}$. Furthermore, observe that condition $mq' = q \pmod{n}$ and (6) give $kq' = q \pmod{n}$, which, in view of the fact that $V$ is a Babbage function of $G$ of order $n$, implies $V^{q'k} = V^q$. Therefore,

$$G^m_{|I_d} = V^q \circ \begin{cases} V^d \circ (G^n)^{b+1} \circ V^{-d}_{|I_d}, & \text{if } i'_d \le k-1, \\ V^d \circ (G^n)^b \circ V^{-d}_{|I_d}, & \text{if } i'_d > k-1 \end{cases} \tag{8}$$

for $d \in \{0, 1, \ldots, n-1\}$.

On the other hand, we may write (5), as follows

$$\beta_d = \begin{cases} m-b-1, & d = 0, \\ -b-1, & d \in \{1, \ldots, n-1\},\ i'_d \le k-1, \\ -b, & d \in \{1, \ldots, n-1\},\ i'_d > k-1. \end{cases}$$

Let $d = 0$, then $i'_0 = 0 \le k-1$ and $b = m - \beta_0 - 1$. Combining these with (8) we obtain

$$G^m_{|I_0} = V^q \circ (G^n_{|I_0})^{b+1} = V^q \circ (G^n_{|I_0})^{m-\beta_0}.$$

Let $d \in \{1, \ldots, n-1\}$. Replacing $b$ by $-\beta_d - 1$ if $i'_d \le k-1$ (resp. by $-\beta_d$ if $i'_d > k-1$) in (8) yields

$$G^m_{|I_d} = V^q \circ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}.$$

Finally,

$$G^m_{|I_d} = V^q \circ \begin{cases} V^d \circ G^{-n\beta_d+mn} \circ V^{-d}_{|I_d}, & d = 0, \\ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}, & d \in \{1, \ldots, n-1\}. \end{cases} \tag{9}$$

Equating (9) with (2) yields for $d \in \{1, \ldots, n-1\}$,

$$T^q_{|I_d} = V^q \circ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}. \tag{10}$$

While, for $d = 0$, we get

$$T^q \circ F^n_{|I_0} = V^q \circ G^{-n\beta_0+nm}_{|I_0}.$$

which, in view of (ii), gives

$$T^q_{|I_0} = V^q \circ G^{-n\beta_0}_{|I_0}. \tag{11}$$

From (10) and (11) we have

$$T^q_{|I_d} = V^q \circ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}, \qquad d \in \{0, \dots, n-1\}. \tag{12}$$

Hence

$$T^q_{|I_{p \,(\mathrm{mod}\, n)}} = V^q \circ V^{p \,(\mathrm{mod}\, n)} \circ G^{-n\beta_{p \,(\mathrm{mod}\, n)}} \circ V^{-p \,(\mathrm{mod}\, n)}_{|I_{p \,(\mathrm{mod}\, n)}}, \qquad p \in \mathbb{N}.$$

As $V^p = V^{p \,(\mathrm{mod}\, n)}$ for $p \in \mathbb{N}$ we obtain

$$T^q_{|I_{p \,(\mathrm{mod}\, n)}} = V^q \circ V^p \circ G^{-n\beta_{p \,(\mathrm{mod}\, n)}} \circ V^{-p}_{|I_{p \,(\mathrm{mod}\, n)}}, \qquad p \in \mathbb{N}. \tag{13}$$

Now let us recall that $T^q[I_d] = I_{(d+q) \,(\mathrm{mod}\, n)}$ for $d \in \{0, \dots, n-1\}$. This, (11) and (13) imply

$$\begin{aligned} T^{lq}_{|I_0} = (T^q)^l_{|I_0} &= \Big(V^q \circ V^{(l-1)q} \circ G^{-n\beta_{q(l-1) \,(\mathrm{mod}\, n)}} \circ V^{-(l-1)q}\Big) \\ &\circ \Big(V^q \circ V^{(l-2)q} G^{-n\beta_{q(l-2) \,(\mathrm{mod}\, n)}} \circ V^{-(l-2)q}\Big) \\ &\circ \dots \circ \big(V^q \circ V^q \circ G^{-n\beta_q} \circ V^{-q}\big) \circ \Big(V^q \circ G^{-n\beta_0}_{|I_0}\Big) \\ &= V^{lq} \circ G^{-n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)}_{|I_0}, \end{aligned}$$

which gives

$$V^{lq}_{|I_0} = T^{lq} \circ G^{n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)}_{|I_0} \tag{14}$$

for $l \in \{1, \dots, n\}$. Now fix $d \in \{1, \dots, n-1\}$. Since $\gcd(q, n) = 1$ there is a unique $l \in \{1, \dots, n\}$ such that $lq = d \,(\mathrm{mod}\, n)$. Hence by (13) we have

$$T^q_{|I_d} = T^q_{|I_{lq \,(\mathrm{mod}\, n)}} = V^q \circ V^{lq} \circ G^{-n\beta_{lq \,(\mathrm{mod}\, n)}} \circ V^{-lq}_{|I_{lq \,(\mathrm{mod}\, n)}}.$$

By substituting (14) twice to the above equation we obtain

$$\begin{aligned} T^q_{|I_d} &= V^q \circ \Big(T^{lq} \circ G^{n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)}\Big) \circ G^{-n\beta_{lq \,(\mathrm{mod}\, n)}} \\ &\circ \Big(G^{-n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)} \circ T^{-lq}_{|I_{lq \,(\mathrm{mod}\, n)}}\Big) \\ &= V^q \circ T^{lq} \circ G^{-n\beta_{lq \,(\mathrm{mod}\, n)}} \circ T^{-lq}_{|I_{lq \,(\mathrm{mod}\, n)}}. \end{aligned}$$

This and the fact that $T$ is a Babbage homeomorphism of $F$ of order $n$, i.e., $T^{lq} = T^{lq \,(\mathrm{mod}\, n)} = T^d$, yield

$$V^q_{|I_d} = T^q \circ T^d \circ G^{n\beta_d} \circ T^{-d}_{|I_d},$$

which in view of (11) completes the proof of (4).

Lemma 3
*Let $u, w \in S^1$, $u \neq w$ and $I := \overrightarrow{[u,w]}$. For every integer $m \geq 2$ and every orientation-preserving homeomorphism $F\colon I \to I$ with $\operatorname{Fix} F \neq \emptyset$ there exist infinitely many orientation-preserving homeomorphisms $G\colon I \to I$ satisfying* (3) *and such that* $\operatorname{Fix} G \neq \emptyset$.

*Proof.* Let $a, b \in \mathbb{R}$ be such that $a < b < a+1$ and $u = e^{2\pi \mathrm{i} a}$ and $w = e^{2\pi \mathrm{i} b}$. Then

$$F\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} f(x)}, \qquad x \in [a,b]$$

for a unique increasing homeomorphism $f\colon [a,b] \to [a,b]$. Clearly, $f$ possesses fixed points. By Theorem 11.2.2 (see [4] ch. 11), there exist infinitely many strictly increasing continuous solutions of

$$g^m(x) = f(x), \qquad x \in [a,b],$$

with $\operatorname{Fix} g \neq \emptyset$. For every such function $g\colon [a,b] \to [a,b]$ define $G\colon I \to I$ by

$$G\left(e^{2\pi \mathrm{i} x}\right) := e^{2\pi \mathrm{i} g(x)}, \qquad x \in [a,b].$$

Then $\operatorname{Fix} G \neq \emptyset$ and

$$G^m\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} g^m(x)} = e^{2\pi \mathrm{i} f(x)} = F(e^{2\pi \mathrm{i} x}), \qquad x \in [a,b].$$

In the proof of the next theorem we will use the following result (see for example [7]).

Lemma 4
*Suppose that $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism, $z \in \operatorname{Per} F$, $\{z, F(z), \ldots, F^{n_F-1}(z)\} = \{z_0, z_1, \ldots, z_{n_F-1}\}$, where $z_0 = z$,*

$$\operatorname{Arg}\frac{z_d}{z_0} < \operatorname{Arg}\frac{z_{d+1}}{z_0} < 2\pi, \qquad d \in \{0, \ldots, n_F - 2\}$$

*and $F(z_0) = z_q$. Then $\alpha(F) = \frac{q}{n_F}$.*

Theorem 2
*Let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and let $m \geq 2$ be an integer such that $\gcd(m, n_F) = 1$. There exists an orientation-preserving homeomorphism $G\colon S^1 \to S^1$ satisfying* (3) *and such that $n_G = n_F$.*
*For every such an $m$ and every $z_0 \in \operatorname{Per} F$, providing that $I_d = I_d(z_0)$ for $d \in \{0, \ldots, n_F - 1\}$ are defined by* (1)*, the solution of* (3) *is of the form:*

$$G(z) := \begin{cases} \left(\Psi^{\operatorname{char} F}\right)^{q'}(H(z)), & z \in I_0, \\ \left(\Psi^{\operatorname{char} F}\right)^{q'}(z), & z \in S^1 \setminus I_0, \end{cases} \tag{15}$$

*where $q' \in \{1, \ldots, n_F - 1\}$ fulfils $mq' = q \pmod{n_F}$, $q := n_F \alpha(F)$, $H\colon I_0 \to I_0$ is an orientation-preserving homeomorphism such that $\operatorname{Fix} H \neq \emptyset$, $H^m = F^{n_F}_{|I_0}$ and*

$$\Psi(z) := T^q \circ T^d \circ H^{\beta_d} \circ T^{-d}(z), \qquad z \in I_d,\ d \in \{0, \ldots, n_F - 1\}, \tag{16}$$

*where $T\colon S^1 \to S^1$ is a Babbage function of $F$ and $\beta_d$ for $d \in \{0,\ldots,n_F-1\}$ are defined by* (5) *with $n = n_F$ and $i'_d$ uniquely determined by $(d+i'_d q') \,(\mathrm{mod}\, n_F) = 0$ for $d \in \{0,\ldots,n_F-1\}$.*

*Proof.* Fix $z_0 \in \mathrm{Per}\, F$ and a mapping $H\colon I_0 \to I_0$ such that $\mathrm{Fix}\, H \neq \emptyset$ and $H^m = F^{n_F}_{|I_0}$ (by Lemma 3 there are infinitely many such mappings). Observe that

$$\Psi[I_d] = T^q \circ T^d \circ H^{\beta_d}[I_0] = T^q[I_d] = I_{(d+q)\,(\mathrm{mod}\, n)}, \qquad d \in \{0,\ldots,n_F-1\}. \quad (17)$$

Moreover, as a composition of orientation-preserving homeomorphisms, $\Psi_{|I_d}$ is an orientation-preserving homeomorphism. Hence $\Psi\colon S^1 \to S^1$ and $G$ are orientation-preserving homeomorphisms.

Now we show that $n_G = n_F$. Put $z_d := F^{d\,\mathrm{char}\, F}(z_0)$ for $d \in \{1,\ldots,n_F-1\}$. Thus by (1), (17) and since $\Psi$ preserves the orientation we get

$$\Psi(z_d) = z_{(d+q)\,(\mathrm{mod}\, n_F)}, \qquad d \in \{0,\ldots,n_F-1\}.$$

This, Lemma 1 and Lemma 4 yield $\alpha(\Psi) = \frac{q}{n_F} = \alpha(F)$ and, in consequence, $n_\Psi = n_F$ and $\mathrm{char}\,\Psi = \mathrm{char}\, F$. Next note that $H(z_0) = z_0$ and $H(z_1) = z_1$. Therefore, by (15) and the definition of $\mathrm{char}\, F$,

$$G(z_d) = \Psi^{q'\,\mathrm{char}\, F}(z_d) = z_{(d+qq'\,\mathrm{char}\, F)\,(\mathrm{mod}\, n_F)} = z_{(d+q')\,(\mathrm{mod}\, n_F)}, \quad (18)$$

for $d \in \{0,\ldots,n_F-1\}$. As $\gcd(q', n_F) = 1$ (see Remark 3) we get $n_F = n_G$.

Our next goal is to prove that $\Psi^{n_F} = \mathrm{id}_{S^1}$. From (17) and (16), in view of the fact that $T^p = T^{p\,(\mathrm{mod}\, n_F)}$ for $p \in \mathbb{N}$, we obtain

$$\begin{aligned}\Psi^{n_F}_{|I_d} &= \left(T^q \circ T^{d+(n_F-1)q} \circ H^{\beta_{(d+(n_F-1)q)\,(\mathrm{mod}\, n_F)}} \circ T^{-d+(n_F-1)q}\right) \\ &\quad \circ \ldots \circ \left(T^q \circ T^{d+q} \circ H^{\beta_{(d+q)\,(\mathrm{mod}\, n_F)}} \circ T^{-d+q}\right) \circ \left(T^q \circ T^d \circ H^{\beta_d} \circ T^{-d}_{|I_d}\right) \\ &= T^q \circ T^{d+(n_F-1)q} \circ H^{\beta_{(d+(n_F-1)q)\,(\mathrm{mod}\, n_F)}+\ldots+\beta_d} \circ T^{-d}_{|I_d}\end{aligned}$$

for $d \in \{0,\ldots,n_F-1\}$. Moreover, since $\gcd(q, n_F) = 1$ we get

$$\{d, (d+q)\,(\mathrm{mod}\, n_F), \ldots, (d+(n_F-1)q)\,(\mathrm{mod}\, n_F)\} = \{0,1,\ldots,n_F-1\}.$$

We thus get

$$\Psi^{n_F}_{|I_d} = T^q \circ T^{d+(n_F-1)q} \circ H^{\beta_{n_F-1}+\ldots+\beta_0} \circ T^{-d}_{|I_d}, \qquad d \in \{0,\ldots,n_F-1\}. \quad (19)$$

Putting $b := \left[\frac{m}{n_F}\right]$ we have (6) with $n = n_F$. By Remark 3 and Remark 1 it follows that the mapping $\{0,\ldots,n_F-1\} \ni d \mapsto i'_d \in \{0,\ldots,n_F-1\}$ is a bijection. Therefore, $i'_d \leq m - bn_F - 1 = k - 1$ holds true for exactly $k$ arguments and one of them is 0, as $i'_0 = 0 \leq k-1$. Hence in view of (5),

$$\beta_{n_F-1} + \ldots + \beta_0 = (n_F - k)(-b) + (k-1)(-b-1) + m - b - 1 = 0.$$

This and (19) give $\Psi^{n_F} = \mathrm{id}_{S^1}$.

What is left is to show that (3) holds. Put $\Psi^{\operatorname{char} F} = V$. By Theorem 1 homeomorphism $V$ is a Babbage function of $G$. Since $q \operatorname{char} F = 1 \pmod{n_F}$ and $\Psi^{n_F} = \mathrm{id}_{S^1}$ we have $\Psi = \Psi^{q \operatorname{char} F} = V^q$. Hence by (16),

$$V^q_{|I_d} = T^q \circ T^d \circ H^{\beta_d} \circ T^{-d}_{|I_d}, \qquad d \in \{0, \ldots, n_F - 1\}. \tag{20}$$

Applying the similar reasoning as in the proof of (iii) of Lemma 2 we obtain

$$T^q_{|I_d} = V^q \circ V^d \circ H^{-\beta_d} \circ V^{-d}_{|I_d}, \qquad d \in \{0, \ldots, n_F - 1\}. \tag{21}$$

Indeed, as $T^p = T^{p \pmod{n_F}}$ for $p \in \mathbb{N}$ from (20) we get

$$V^q_{|I_{p \pmod{n_F}}} = T^q \circ T^p \circ H^{\beta_{p \pmod{n_F}}} \circ T^{-p}_{|I_{p \pmod{n_F}}}, \qquad p \in \mathbb{N}. \tag{22}$$

Thus

$$V^{lq}_{|I_0} = T^{lq} \circ H^{\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \ldots + \beta_q + \beta_0\right)}_{|I_0},$$

which gives

$$T^{lq}_{|I_0} = V^{lq} \circ H^{-\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \ldots + \beta_q + \beta_0\right)}_{|I_0} \tag{23}$$

for $l \in \{1, \ldots, n_F\}$. Now fix $d \in \{1, \ldots, n_F - 1\}$. Since $\gcd(q, n_F) = 1$ there is a unique $l \in \{1, \ldots, n_F\}$ such that $lq = d \pmod{n_F}$. Hence by (22) we have

$$V^q_{|I_d} = V^q_{|I_{lq \pmod{n_F}}} = T^q \circ T^{lq} \circ H^{\beta_{lq \pmod{n_F}}} \circ T^{-lq}_{|I_{lq \pmod{n_F}}}.$$

By substituting (23) twice to the above equation we obtain

$$\begin{aligned} V^q_{|I_d} &= T^q \circ \left(V^{lq} \circ H^{-\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \ldots + \beta_q + \beta_0\right)}\right) \circ H^{\beta_{lq \pmod{n_F}}} \\ &\quad \circ \left(H^{\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \ldots + \beta_q + \beta_0\right)} \circ V^{-lq}_{|I_{lq \pmod{n_F}}}\right) \\ &= T^q \circ V^{lq} \circ G^{\beta_{lq \pmod{n_F}}} \circ V^{-lq}_{|I_{lq \pmod{n_F}}}. \end{aligned}$$

This and the fact that $V$ is a Babbage homeomorphism of $G$ of order $n_F$, i.e., $V^{lq} = V^{lq \pmod{n_F}} = T^d$, yield (21).

Now observe that from (2) and (21), since $H^m = F^{n_F}_{|I_0}$ and $kq' = q \pmod{n_F}$, we get

$$F_{|I_d} = V^{kq'} \circ \begin{cases} V^d \circ H^{-\beta_d + m} \circ V^{-d}_{|I_d}, & d = 0, \\ V^d \circ H^{-\beta_d} \circ V^{-d}_{|I_d}, & d \in \{1, \ldots, n_F - 1\}, \end{cases}$$

which in view of (15), (6) and Theorem 1 gives $F = G^m$.

We finish with the following observations

Remark 4

If the assumptions of Theorem 2 are fulfilled, then

(i) from (18), Lemma 4, Lemma 1 it follows that $\alpha(G) = \frac{q'}{n_F}$,

(ii) by Lemma 3 there are infinitely many solutions of (3) with $n_G = n_F$,

(iii) Lemma 2 and Theorem 2 imply that every orientation-preserving continuous solution of (3) with $n_G = n_F$ is given by (15) and (16).

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# Bounding symbolic powers via asymptotic multiplier ideals

**Abstract.** We revisit a bound on symbolic powers found by Ein–Lazarsfeld–Smith and subsequently improved by Takagi–Yoshida. We show that the original argument of [6] actually gives the same improvement. On the other hand, we show by examples that any further improvement based on the same technique appears unlikely. This is primarily an exposition; only some examples and remarks might be new.

## 1. Uniform bounds for symbolic powers

For a radical ideal $I$, the symbolic power $I^{(p)}$ is the collection of elements that vanish to order at least $p$ at each point of Zeros$(I)$. If $I$ is actually prime, then $I^{(p)}$ is the $I$-associated primary component of $I^p$; if $I$ is only radical, writing $I = C_1 \cap \ldots \cap C_s$ as an intersection of prime ideals, $I^{(p)} = C_1^{(p)} \cap \ldots \cap C_s^{(p)}$. The inclusion $I^p \subseteq I^{(p)}$ always holds, but the reverse inclusion holds only in some special cases, such as when $I$ is a complete intersection.

Swanson [15] showed that for rings $R$ satisfying a certain hypothesis, for each ideal $I$, there is an integer $e = e(I)$ such that the symbolic power $I^{(er)} \subseteq I^r$ for all $r \geq 0$. Ein–Lazarsfeld–Smith [6] showed that in a regular local ring $R$ in equal characteristic 0 and for $I$ a radical ideal, one can take $e(I) = \mathrm{bight}(I)$, the *big height* of $I$, which is the maximum of the codimensions of the irreducible components of the closed subset of zeros of $I$. In particular, $\mathrm{bight}(I)$ is at most the dimension of the ambient space, so $e = \dim R$ is a single value that works for all ideals. More generally, for any $k \geq 0$, $I^{(er+kr)} \subseteq (I^{(k+1)})^r$ for all $r \geq 1$. Very shortly thereafter, Hochster–Huneke [9] generalized this result by characteristic $p$ methods.

It is natural to regard these results in the form $I^{(m)} \subseteq I^r$ for $m \geq f(r) = er$, $e = \mathrm{bight}(I)$. Replacing $f(r) = er$ with a smaller function would give a stronger bound on symbolic powers (containment in $I^r$ would begin sooner). So it is natural to ask, how far can one reduce the bounding function $f(r) = er$?

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Bocci–Harbourne [3] introduced the *resurgence* of $I$, $\rho(I) = \sup\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\}$. Thus if $m > \rho(I)r$, $I^{(m)} \subseteq I^r$. The Ein–Lazarsfeld–Smith and Hochster–Huneke results show $\rho(I) \leq \operatorname{bight}(I) \leq \dim R$. It can be smaller. For example, if $I$ is smooth or a reduced complete intersection, $\rho(I) = 1$. More interestingly, Bocci–Harbourne [3] show that if $I$ is an ideal of $n$ reduced points in general position in $\mathbb{P}^2$, $\rho(I) = \rho_n \leq \frac{3}{2}$. On the other hand, Bocci–Harbourne show for each $n$, $1 \leq e \leq n$, and $\epsilon > 0$ there are ideals $I$ on $\mathbb{P}^n$ with $\operatorname{bight}(I) = e$ such that $\rho(I) > e - \epsilon$. This suggests that one cannot expect improvement in the slope of the linear bound $m \geq er$, at least not in very general terms. So one naturally turns toward the possibility of subtracting a constant term.

Huneke raised the question of whether, for $I$ an ideal of reduced points in $\mathbb{P}^2$, $I^{(3)} \subseteq I^2$. Bocci–Harbourne's result $\rho \leq \frac{3}{2}$ gives an affirmative answer to Huneke's question, and much more, for points in general position. Some other cases have been treated, e.g., points on a conic, but the general case, i.e., points in arbitrary position, remains open.

A conjecture of Harbourne (Conjecture 8.4.3 in [1]) states that for a homogeneous ideal $I$ on $\mathbb{P}^n$, $I^{(m)} \subseteq I^r$ for all $m \geq nr - (n-1)$, and even stronger, that the containment holds for all $m \geq er - (e-1)$, where $e = \operatorname{bight}(I)$. Huneke's question would follow at once as the case $n = e = r = 2$.

Some results in this direction have been obtained by various authors. Huneke has observed that Harbourne's conjecture holds in characteristic $p$ for values $r = p^k$, $k \geq 1$, see Example IV.5.3 of [8] or Example 8.4.4 of [1]. Takagi–Yoshida [17] and Hochster–Huneke independently showed by characteristic $p$ methods that $I^{(er+kr-1)} \subseteq (I^{(k+1)})^r$ for all $k \geq 0$ and $r \geq 1$ when $I$ is $F$-pure (see below). More generally, Takagi–Yoshida show a characteristic $p$ version of the following:

Theorem 1.1 ([17])
*Let $R$ be a regular local ring of equal characteristic 0, $I \subseteq R$ a reduced ideal, $e = \operatorname{bight}(I)$ the greatest height of an associated prime of $I$, and $\ell$ an integer, $0 \leq \ell < \operatorname{lct}(I^{(\bullet)})$, where $\operatorname{lct}(I^{(\bullet)})$ is the log canonical threshold of the graded system of symbolic powers of $I$, see below. Then $I^{(m)} \subseteq I^r$ whenever $m \geq er - \ell$. More generally, for any $k \geq 0$, $I^{(m)} \subseteq (I^{(k+1)})^r$ whenever $m \geq er + kr - \ell$.*

This statement is a slight modification of Remark 3.4 of [17].

The Ein–Lazarsfeld–Smith uniform bounds on symbolic powers described above are the case $\ell = 0$. The $F$-pure case implies $\operatorname{lct}(I^{(\bullet)}) > 1$, so we may take $\ell = 1$. (More precisely, $F$-pure means $\operatorname{lct}(I) > 1$, and we will see $\operatorname{lct}(I^{(\bullet)}) \geq \operatorname{lct}(I)$.)

The idea of the proof is to produce an ideal $J$ with $I^{(m)} \subseteq J$ and $J \subseteq (I^{(k+1)})^r$. Ein–Lazarsfeld–Smith introduced asymptotic multiplier ideals in [6] and, among other results, proved the uniform bounds described above by taking $J$ to be an asymptotic multiplier ideal. For Takagi–Yoshida the ideal $J$ is a generalized test ideal, a characteristic $p$ analogue of the asymptotic multiplier ideals introduced by Hara–Yoshida [11].

In this note, $J$ will be an asymptotic multiplier ideal. We will review multiplier ideals in §2 and discuss some examples in §3: the asymptotic multiplier ideals of monomial ideals and hyperplane arrangements. We will revisit the argument given by Ein–Lazarsfeld–Smith in the case $\ell = 0$ to show that it actually gives Theorem 1.1 in §4.

In §5 we consider two ways in which the argument of §4 falls short of the improved bounds we hope for. First, the condition $0 \le \ell < \operatorname{lct}(I^{(\bullet)})$, while generalizing the result of [6], is nevertheless quite restrictive. Second, the argument of [6] actually produces two ideals, $I^{(m)} \subseteq J_1 \subseteq J_2 \subseteq (I^{(k+1)})^r$. We will consider as an example the ideal $I = (xy, xz, yz)$ of the union of the three coordinate axes in $\mathbb{C}^3$. We will show that in this example the first and last inclusions are actually equalities, while the middle inclusion $J_1 \subseteq J_2$ is very far. So if any improvement remains to be found, one must consider the middle inclusion.

## 2. Multiplier ideals

Henceforth we fix $X = \mathbb{C}^n$ and consider ideals in the ring $R = \mathbb{C}[x_1, \ldots, x_n]$.

Note that for a prime homogeneous ideal $I$, a homogeneous form $F$ vanishes to order $p$ along the projective variety defined by $I$ in $\mathbb{P}^{n-1}$ if and ony if it vanishes to order $p$ on the affine variety defined by $I$ in $\mathbb{C}^n$. In this way the Bocci–Harbourne results and Huneke question for points in $\mathbb{P}^2$ translate to questions about symbolic powers of (homogeneous) ideals in the affine setting.

### 2.1. Ordinary multiplier ideals

To an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, regarded as a sheaf of ideals on $X = \mathbb{C}^n$, and a real parameter $t \ge 0$ one may associate the *multiplier ideal* $\mathcal{J}(I^t) \subseteq \mathbb{C}[x_1, \ldots, x_n]$. The multiplier ideals are defined in terms of a resolution of singularities of $I$. For details, see [4], [12].

Note, in the notation $\mathcal{J}(I^t)$ the $t$ indicates dependance on the parameter $t$, rather than a power of $I$. In particular, $\mathcal{J}(I^t)$ is defined for all real $t \ge 0$, whereas $I^t$ on its own only makes sense for integer $t \ge 0$. See, however, Property 2.2.

Rather than present the somewhat involved definition here, we give a short list of properties of multiplier ideals which are all that we will use. (The reader may take these as axioms, although the properties listed here do not characterize multiplier ideals.)

Property 2.1
For any nonzero ideal $I$, $\mathcal{J}(I^0) = (1)$, the unit ideal. As the parameter $t$ increases, the multiplier ideals get smaller: if $t_1 < t_2$, then $\mathcal{J}(I^{t_1}) \supseteq \mathcal{J}(I^{t_2})$.
On the other hand, if $I_1 \subseteq I_2$, then $\mathcal{J}(I_1^t) \subseteq \mathcal{J}(I_2^t)$.
Thus multiplier ideals, as functions of two arguments, are "order-preserving" in the ideal and "order-reversing" in the real parameter.

Property 2.2
For any real number $t \ge 0$ and integer $k > 0$, $\mathcal{J}(I^{kt}) = \mathcal{J}((I^k)^t)$.

Property 2.3
For any $t \ge 0$ and integer $p \ge 0$, $I^p \mathcal{J}(I^t) \subseteq \mathcal{J}(I^{p+t})$. See Proposition 9.2.32 (iv) of [12].

Property 2.4
When $\mathrm{Zeros}(I)$ is smooth and irreducible with codimension $\operatorname{codim}(\mathrm{Zeros}(I)) = e = \mathrm{bight}(I)$, $\mathcal{J}(I^t) = I^{\lfloor t \rfloor - e + 1}$. In particular, $\mathcal{J}(I^t) \subseteq I$ for $t \geq e$. More generally, if $I$ is reduced and $\mathrm{Zeros}(I) = V_1 \cup \ldots \cup V_s$, then restricting to a neighborhood of a general point on each $V_i$, we see $\mathcal{J}(I^t)$ vanishes on $V_i$ for $t \geq \operatorname{codim} V_i$, hence $\mathcal{J}(I^t) \subseteq I$ for $t \geq \max \operatorname{codim} V_i = \mathrm{bight}(I)$.

The above list is a small selection of the many interesting properties of multiplier ideals. See [4], [12] for more, including excellent expositions of the definition (from which all the above properties follow immediately). Among these many other properties we single out one which we will use here, due to Demailly–Ein–Lazarsfeld [5].

Subadditivity Theorem
$\mathcal{J}(I^{t_1+t_2}) \subseteq \mathcal{J}(I^{t_1})\mathcal{J}(I^{t_2})$. *In particular for any integer* $r \geq 0$, $\mathcal{J}(I^{rt}) \subseteq \mathcal{J}(I^t)^r$.

## 2.2. Asymptotic multiplier ideals

A *graded system* of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}_{n=1}^{\infty}$ is a collection of ideals satisfying $\mathfrak{a}_p \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$, and (to avoid trivialities) at most finitely many of the $\mathfrak{a}_n$ may be zero. Note that $\mathfrak{a}_p$, $\mathfrak{a}_{p+1}$ are not required to satisfy any particular relation, but $(\mathfrak{a}_p)^k \subseteq \mathfrak{a}_{kp}$. By convention, $\mathfrak{a}_0 = \mathbb{C}[x_1, \ldots, x_n]$, so that $\bigoplus_{n=0}^{\infty} \mathfrak{a}_n$ is a $\mathbb{C}[x_1, \ldots, x_n]$-algebra. A trivial graded system is one of the form $\mathfrak{a}_n = \mathfrak{a}_1^n$. Our main interest will be in the graded system of symbolic powers of a (reduced) ideal $I$, $I^{(\bullet)} = \{I^{(n)}\}_{n \geq 0}$.

To a graded system $\mathfrak{a}_\bullet$ and real parameter $t \geq 0$ one can associate an *asymptotic multiplier ideal* $\mathcal{J}(\mathfrak{a}_\bullet^t)$, or $\mathcal{J}(t \cdot I^{(\bullet)})$, defined by

$$\mathcal{J}(\mathfrak{a}_\bullet^t) = \max_{p \geq 1} \mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big).$$

This definition was given in [6]. We must justify the existence and well-definedness of this maximum; we repeat the argument of [6]. Note that since $(\mathfrak{a}_p)^q \subseteq \mathfrak{a}_{qp}$, by the properties of multiplier ideals we have

$$\mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big) = \mathcal{J}\big((\mathfrak{a}_p^q)^{\frac{t}{pq}}\big) \subseteq \mathcal{J}\big(\mathfrak{a}_{pq}^{\frac{t}{pq}}\big).$$

The Noetherian property assures that among the ideals $\mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big)$, one is a (relative) maximum. If $\mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big)$ is a maximum, then by the above, $\mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big) = \mathcal{J}\big(\mathfrak{a}_{pq}^{\frac{t}{pq}}\big)$. Hence if $\mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big)$ and $\mathcal{J}\big(\mathfrak{a}_q^{\frac{t}{q}}\big)$ both are maxima, then they coincide with each other. Thus there is a unique maximum of this collection of ideals.

In particular, $\mathcal{J}(\mathfrak{a}_\bullet^t) = \mathcal{J}\big(\mathfrak{a}_p^{\frac{t}{p}}\big)$ for $p \gg 0$ and sufficiently divisible, i.e., for all sufficiently large multiples of some $p_0$. We say that such a $p$ *computes the asymptotic multiplier ideal.*

Example 2.5
In the trivial case $\mathfrak{a}_n = \mathfrak{a}_1^n$, the asymptotic multiplier ideals reduce to the ordinary multiplier ideals: $\mathcal{J}(\mathfrak{a}_\bullet^t) = \mathcal{J}(\mathfrak{a}_1^t)$. This has the following consequence: If $I$ is a

reduced ideal defining a smooth and irreducible variety of codimension $e$, then

$$\mathcal{J}(t \cdot I^{(\bullet)}) = \mathcal{J}(I^t) = I^{\lfloor t \rfloor + 1 - e}.$$

As before, if $I$ is only reduced, then by restricting to a neighborhood of a smooth point on each irreducible component of Zeros$(I)$, we see that $\mathcal{J}(t \cdot I^{(\bullet)}) \subseteq I$ for $t \geq e = \text{bight}(I)$. And, more generally, $\mathcal{J}((e+k) \cdot I^{(\bullet)}) \subseteq I^{(k+1)}$ for any $k \geq 0$ and any reduced ideal $I$.

Remark 2.6
Conversely, $\mathfrak{a}_n \subseteq \mathcal{J}(\mathfrak{a}_\bullet^n)$. In fact, for every $n$, $t$, $\mathfrak{a}_n \cdot \mathcal{J}(\mathfrak{a}_\bullet^t) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{t+n})$ (Theorem 11.1.19 (iii) of [12]). This is exactly the extra piece we will add to the argument of [6] to deduce Theorem 1.1.

As before, $\mathcal{J}(\mathfrak{a}_\bullet^0) = (1)$ and if $t_1 < t_2$, then $\mathcal{J}(\mathfrak{a}_\bullet^{t_1}) \supseteq \mathcal{J}(\mathfrak{a}_\bullet^{t_2})$. The asymptotic multiplier ideals satisfy subadditivity: $\mathcal{J}(\mathfrak{a}_\bullet^{t_1+t_2}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{t_1})\mathcal{J}(\mathfrak{a}_\bullet^{t_2})$, so $\mathcal{J}(\mathfrak{a}_\bullet^{rt}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^t)^r$ [12, 11.2.3]. This follows immediately from the subadditivity theorem for ordinary multiplier ideals. (Let $p$ large and divisible enough compute all the asymptotic multiplier ideals appearing in the equation, then apply the ordinary subadditivity theorem for $\mathfrak{a}_p$.)

### 2.3. Log canonical thresholds

For an ideal $I \neq (0), (1)$, we define $\text{lct}(I) = \sup\{t \mid \mathcal{J}(I^t) = (1)\}$. This is a positive rational number. It turns out that $\mathcal{J}(I^{\text{lct}(I)}) \neq (1)$. (See [7] or [12].)

Let $I$ be a radical ideal and let $e'$ be the minimum of the codimensions of the irreducible components of Zeros$(I)$. Then lct$(I)$ satisfies

$$0 < \text{lct}(I) \leq e'.$$

(Restricting to a neighborhood of a general point on a codimension $e'$ component of Zeros$(I)$, $\mathcal{J}(I^{e'})$ vanishes on the component by Property 2.4.)

For a graded system of ideals $\mathfrak{a}_\bullet$, we define $\text{lct}(\mathfrak{a}_\bullet) = \sup\{t \mid \mathcal{J}(\mathfrak{a}_\bullet^t) = (1)\}$. This may be infinite or irrational. However for the graded system of symbolic powers of a radical ideal $I$, we have $\text{lct}(I^{(\bullet)}) \leq e'$ as above.

As shown in [13, Remark 3.3],

$$\text{lct}(\mathfrak{a}_\bullet) = \sup p \, \text{lct}(\mathfrak{a}_p) = \lim p \, \text{lct}(\mathfrak{a}_p).$$

Taking $p = 1$, this shows $\text{lct}(I^{(\bullet)}) \geq \text{lct}(I)$ for a radical ideal $I$.

## 3. Examples

In this section we give the asymptotic multiplier ideals of graded systems of monomial ideals, especially for the symbolic powers of a radical (i.e., squarefree) monomial ideal, and the asymptotic multiplier ideals of graded systems of divisor and hyperplane arrangements.

### 3.1. Monomial ideals

The following theorem gives the ordinary multiplier ideals of a monomial ideal.

THEOREM 3.1 ([10])
*Let $I$ be a monomial ideal with Newton polyhedron $N = \mathrm{Newt}(I)$. Then $\mathcal{J}(I^t)$ is the monomial ideal containing $x^v$ if and only if $v + (1,\dots,1) \in \mathrm{Int}(t \cdot N)$.*

Here $\mathrm{Int}(\,)$ denotes topological interior. In particular, $\mathrm{lct}(I) = \frac{1}{t}$, where $t \cdot (1,\dots,1)$ is in the boundary of $\mathrm{Newt}(I)$.

Let $I_\bullet = \{I_p\}$ be a graded system of monomial ideals. Let $N_p = \mathrm{Newt}(I_p)$. Then $I_p^k \subseteq I_{pk}$, so $k \cdot N_p \subseteq N_{pk}$, which means $\frac{1}{p} N_p \subseteq \frac{1}{pk} N_{pk}$. Let $N(I_\bullet) = \bigcup \frac{1}{p} N_p$. Since this is an ascending union of convex sets, it is convex.

THEOREM 3.2 ([13])
*$\mathcal{J}(I_\bullet^t)$ is the monomial ideal containing $x^v$ if and only if $v + (1,\dots,1) \in \mathrm{Int}(t \cdot N(I_\bullet))$.*

*Proof.* If $p$ computes $\mathcal{J}(I_\bullet^t)$ and $x^v \in \mathcal{J}(I_\bullet^t) = \mathcal{J}\big(I_p^{\frac{t}{p}}\big)$, then $v + (1,\dots,1) \in \mathrm{Int}(\frac{t}{p} N_p) \subseteq \mathrm{Int}(t \cdot N(I_\bullet))$. Conversely if $v + (1,\dots,1) \in \mathrm{Int}(t \cdot N(I_\bullet))$, then $v + (1,\dots,1) \in \mathrm{Int}(\frac{t}{p} N_p)$ for some $p$, whence $x^v \in \mathcal{J}\big(I_p^{\frac{t}{p}}\big) \subseteq \mathcal{J}(I_\bullet^t)$.

For a graded system of monomial ideals, $\mathrm{lct}(I_\bullet) = \frac{1}{t}$, where $t \cdot (1,\dots,1)$ is in the boundary of $N(I_\bullet)$.

More can be said in a special situation:

PROPOSITION 3.3
*If a graded system is given by $I_p = C_1^p \cap \ldots \cap C_r^p$ for fixed monomial ideals $C_1,\dots,C_r$, then in the above notation, $N(I_\bullet) = \bigcap \mathrm{Newt}(C_i)$.*

*Proof.* For a monomial ideal $\mathfrak{a}$, let $\mathrm{monom}(\mathfrak{a})$ denote the set of exponent vectors of monomials in $\mathfrak{a}$, so that $\mathrm{Newt}(\mathfrak{a})$ is the convex hull $\mathrm{conv}(\mathrm{monom}(\mathfrak{a}))$. For $p \geq 1$ we have $\mathrm{monom}(I_p) = \bigcap \mathrm{monom}(C_i^p)$, so

$$\mathrm{Newt}(I_p) \subseteq \bigcap \mathrm{Newt}(C_i^p) = p \cdot \bigcap \mathrm{Newt}(C_i).$$

This shows $N(I_\bullet) \subseteq \bigcap \mathrm{Newt}(C_i)$.

For the reverse inclusion, note $\bigcap \mathrm{Newt}(C_i)$ is a rational polyhedron. For $p$ sufficiently divisible, $p \cdot \bigcap \mathrm{Newt}(C_i)$ is a lattice polyhedron; in particular all its extremal points (vertices) have integer coordinates, and $p \cdot \bigcap \mathrm{Newt}(C_i)$ is the convex hull of the integer (lattice) points it contains. So let $v$ be an integer point in $p \cdot \bigcap \mathrm{Newt}(C_i) = \bigcap \mathrm{Newt}(C_i^p)$. Then $x^v \in C_i^p$ for each $i$, so $x^v \in \bigcap C_i^p = I_p$. This shows $p \cdot \bigcap \mathrm{Newt}(C_i) \subseteq \mathrm{conv}(\mathrm{monom}(I_p))$. Therefore $\bigcap \mathrm{Newt}(C_i) \subseteq \frac{1}{p} \mathrm{Newt}(I_p) \subseteq N(I_\bullet)$.

One can check that in the situation of the above proposition, $\mathrm{lct}(I_\bullet) = \min \mathrm{lct}(C_i)$.

Proposition 3.4
*Let $I = I_1$ be a reduced monomial ideal and $I_p = I^{(p)}$. Suppose $I$ is not the maximal ideal. Let $N$ be the convex region defined by the linear inequalities that correspond to unbounded facets of* $\operatorname{Newt}(I)$. *Then $N = N(I^{(\bullet)})$; in particular $\mathcal{J}(t \cdot I^{(\bullet)})$ is the monomial ideal containing $x^v$ if and only if $v + (1, \dots, 1) \in \operatorname{Int}(t \cdot N)$.*

*Proof.* Let $I = C_1 \cap \ldots \cap C_r$, the $C_i$ minimal primes of $I$. Then $I^{(p)} = C_1^p \cap \ldots \cap C_r^p$. As long as $I$ is non-maximal, equivalently each $C_i$ is non-maximal, the $\operatorname{Newt}(C_i)$, together with the facets of the positive orthant, correspond precisely to the unbounded facets of $\operatorname{Newt}(I)$. The result follows by the previous propositions.

In particular, each $\operatorname{lct}(C_i) = \operatorname{ht} C_i$, so $\operatorname{lct}(I^{(\bullet)}) = \min \operatorname{ht} C_i = e'$, where $e'$ is the minimum codimension of any irreducible component of the variety $V(I)$.

## 3.2. Hyperplane arrangements

Let $D$ be a divisor with real (or rational or integer) coefficients. The multiplier ideals $\mathcal{J}(t \cdot D)$ are defined similarly to the multiplier ideals of ideals. All the properties described above hold for multiplier ideals of divisors. In fact, when $D$ is a divisor with integer coefficients with defining ideal $I$, $\mathcal{J}(t \cdot D) = \mathcal{J}(I^t)$. See [12] for details.

The multiplier ideals of hyperplane arrangements were computed in [14], with the following result.

Theorem 3.5
*Let $D = b_1 H_1 + \ldots + b_r H_r$ be a weighted central arrangement, where the $H_i$ are hyperplanes in $\mathbb{C}^n$ containing the origin and the $b_i$ are nonnegative real numbers, the weights. Let $L(D)$ be the intersection lattice of the arrangement $D$, the set of proper subspaces of $\mathbb{C}^n$ which are intersections of the $H_i$. For $W \in L(D)$, let $r(W) = \operatorname{codim}(W)$ and $s(W) = \sum\{b_i \mid W \subseteq H_i\} = \operatorname{ord}_W(D)$. Then the multiplier ideals of $D$ are given by*

$$\mathcal{J}(t \cdot D) = \bigcap_{W \in L(D)} I_W^{\lfloor t \cdot s(W) \rfloor + 1 - r(W)},$$

*where $I_W$ is the ideal of $W$.*

In fact, the intersection over $W \in L(D)$ can be reduced to an intersection over $W \in \mathcal{G}$ for certain subsets $\mathcal{G} \subseteq L(D)$ called building sets; see [16]. The log canonical threshold is given by $\operatorname{lct}(D) = \min_{W \in L(D)} \frac{r(W)}{s(W)}$; this may be reduced to a minimum over $W \in \mathcal{G}$.

With this in hand it is easy to describe a similar result for graded systems of hyperplane arrangements.

We will say a *graded system of divisors* is a sequence $D_\bullet = \{D_p\}_{p \geq 1}$ such that $D_p + D_q \geq D_{p+q}$. Equivalently, for each component $E$, the $\operatorname{ord}_E(D_p)$ satisfy $\operatorname{ord}_E(D_p) + \operatorname{ord}_E(D_q) \geq \operatorname{ord}_E(D_{p+q})$. If the $D_p$ have integer weights, then the condition of the $D_p$ forming a graded system of divisors is equivalent to requiring the ideals $I_p = I(D_p)$ to form a graded system of ideals. Define the asymptotic multiplier ideal $\mathcal{J}(t \cdot D_\bullet) = \max_p \mathcal{J}(\frac{t}{p} D_p)$, as for graded systems of ideals.

The following lemma will be helpful:

Lemma 3.6 ([13], Lemma 1.4)
*Let $\{a_p\}$ be a sequence of non-negative real numbers such that $a_p + a_q \geq a_{p+q}$ for all $p, q$. Then $\frac{1}{p} a_p$ converges to a finite limit; in fact $\frac{1}{p} a_p \to \inf \frac{1}{p} a_p$.*

For a graded system $D_\bullet$ of divisors, let

$$D_\infty = \sum a_E E, \quad \text{where } a_E = \lim_{p\to\infty} \frac{1}{p} \mathrm{ord}_E(D_p).$$

Proposition 3.7
*Let $D_\bullet$ be a graded system of divisors. Then $\mathcal{J}(t \cdot D_\bullet) = \mathcal{J}(t \cdot D_\infty)$.*

This follows from considering a common resolution of singularities of all the $D_p$ and $D_\infty$. The following is an immediate consequence.

Proposition 3.8
*Let $D_\bullet$ be a graded system of divisors, where each $D_p$ is a central hyperplane arrangement. Let the hyperplanes be $H_1, \ldots, H_r$. Let $D_p = b_{1,p} H_1 + \ldots + b_{r,p} H_r$, and let $b_{i,\infty} = \lim \frac{b_{i,p}}{p}$. Let $L(D_0)$ be the intersection lattice of the (reduced) arrangement $D_0 = H_1 \cup \ldots \cup H_r$, and for $W \in L(D_0)$ let $s_\infty(W) = \sum\{b_{i,\infty} : W \subseteq H_i\}$, $r(W) = \mathrm{codim}(W)$. Then*

$$\mathcal{J}(t \cdot D_\bullet) = \bigcap_{W \in L(D_0)} I_W^{\lfloor t \cdot s_\infty(W) \rfloor + 1 - r(W)} = \mathcal{J}(t \cdot D_\infty),$$

*where $D_\infty$ is defined as above.*

Again the intersection can be reduced to $W \in \mathcal{G}$ for a building set $\mathcal{G} \subseteq L(D_0)$. The log canonical threshold is given by $\mathrm{lct}(D_\bullet) = \mathrm{lct}(D_\infty) = \min_W \frac{r(W)}{s_\infty(W)}$.

## 4. Proof of Theorem

At this point the theorem is easy to prove. The real work was to develop the definition of multiplier ideals and show they have the properties described in §2.

We have $\mathcal{J}(I^e) \subseteq I$. Together with the subadditivity theorem this gives the following chain of inclusions:

$$\mathcal{J}(I^{er}) \subseteq \mathcal{J}(I^e)^r \subseteq I^r.$$

Unfortunately $I^{(er)}$ is not necessarily contained in $\mathcal{J}(I^{er})$. We must enlarge these multiplier ideals enough to contain $I^{(er)}$ but not too much to destroy the containment in $I^r$. First rewrite the above as

$$\mathcal{J}((I^p)^{\frac{er}{p}}) \subseteq \mathcal{J}((I^p)^{\frac{e}{p}})^r \subseteq I^r.$$

These are the same ideals by Property 2.2. Now let $p$ be sufficiently large and divisible and enlarge $I^p$ to $I^{(p)}$. The multiplier ideals become asymptotic multiplier ideals, and we will see in a moment that the inclusions above still hold:

$$\mathcal{J}(er \cdot I^{(\bullet)}) \subseteq \mathcal{J}(e \cdot I^{(\bullet)})^r \subseteq I^r.$$

By Remark 2.6 we have $I^{(er)} \subseteq \mathcal{J}(er \cdot I^{(\bullet)})$. So this shows $I^{(er)} \subseteq I^r$. This explains why we use asymptotic multiplier ideals rather than ordinary multiplier ideals in this proof. We arrive at the following proof of Theorem 1.1.

*Proof.* We have the following chain of inclusions:

$$\begin{aligned} I^{(er+kr-\ell)} &= I^{(er+kr-\ell)}\mathcal{J}(\ell \cdot I^{(\bullet)}) \\ &\subseteq \mathcal{J}((er+kr) \cdot I^{(\bullet)}) \subseteq \mathcal{J}((e+k) \cdot I^{(\bullet)})^r \\ &\subseteq (I^{(k+1)})^r \end{aligned} \tag{$\star$}$$

which is justified as follows. For $\ell < \operatorname{lct}(I^{(\bullet)})$, $\mathcal{J}(\ell \cdot I^{(\bullet)}) = (1)$. The first inclusion is Remark 2.6. The second inclusion holds by the subadditivity theorem. The last inclusion is Example 2.5.

Theorem 2.2 of [6] is shown by exactly the above argument with $\ell = 0$.

## 5. Non-improvement

Using "classical" methods, Bocci–Harbourne have given some improvements in special cases to the Ein–Lazarsfeld–Smith theorem that $I^{(er)} \subseteq I^r$ for every reduced ideal $I$ with $\operatorname{bight}(I) = e$. For example [3] shows the resurgence of an ideal $I$ of general points in $\mathbb{P}^2$ is at most $\frac{3}{2}$, so $I^{(m)} \subseteq I^r$ for $m \geq \frac{3r}{2}$. However, the argument given above for the proof of Theorem 1.1, either via asymptotic multiplier ideals or via characteristic $p$ methods, is the only way I am aware of to show for every reduced ideal $I$ of height $e$ that $I^{(er)} \subseteq I^r$ (i.e., the resurgence is at most $e$) or even that the resurgence is finite for every reduced ideal.

One may ask, how far can the same multiplier ideal methods be pushed to improve the bounds in the Ein–Lazarsfeld–Smith theorem?

### 5.1. Restriction of log canonical threshold

The value $\ell$ in Theorem 1.1 is severely restricted. Let $e'$ be the minimum of the codimensions of the irreducible components of $\operatorname{Zeros}(I)$. We saw $0 < \operatorname{lct}(I) \leq e'$, but it often happens that $\operatorname{lct}(I)$ is much smaller than $e'$. For $I$ a homogeneous ideal in $\mathbb{C}[x_1, \dots, x_n]$, we have

$$\frac{1}{\operatorname{mult}_0(I)} \leq \operatorname{lct}(I) \leq \frac{n}{\operatorname{mult}_0(I)}$$

([12, 9.3.2-3]), where $\operatorname{mult}_0(I)$ is the multiplicity of $I$ at the origin, equivalently, the least degree of a nonzero form in $I$. So if $\operatorname{lct}(I) > 1$, then $I$ must contain a form of degree strictly less than $n$.

For ideals of reduced sets of points in $\mathbb{P}^2$ one can show the converse, so $\operatorname{lct}(I) > 1$ if and only if the points lie on a conic (which may be reducible). So Theorem 1.1 implies Harbourne's conjecture and answers Huneke's question only for points on a conic, which (for smooth conics at least) had already been treated by Bocci–Harbourne [2].

We only need $\ell < \operatorname{lct}(I^{(\bullet)})$, which is a priori less restrictive than $\ell < \operatorname{lct}(I)$, but still restricts us to $\ell \leq e' - 1$. Indeed, there are radical ideals $I$ with $\operatorname{lct}(I) < \operatorname{lct}(I^{(\bullet)})$. However I do not know of an ideal $I$ such that there is an integer $\ell$, $\operatorname{lct}(I) \leq \ell < \operatorname{lct}(I^{(\bullet)})$.

For a radical homogeneous ideal $I$,

$$\operatorname{lct}(I^{(\bullet)}) \leq \frac{n}{\lim\limits_{p\to\infty} \frac{1}{p} \operatorname{mult}_0(I^{(p)})},$$

where the limit exists because $\operatorname{mult}_0(I^{(p)}) + \operatorname{mult}_0(I^{(q)}) \geq \operatorname{mult}_0(I^{(p+q)})$. If $\operatorname{lct}(I^{(\bullet)}) > 1$, then for some $p$ there must be a homogeneous form $F$ vanishing to order $p$ along the variety defined by $I$, of degree strictly less than $pn$. This is weaker than the requirement that if $\operatorname{lct}(I) > 1$, then $I$ must contain a form of degree less than $n$, which is the same statement with the added condition $p = 1$; but it does not seem very much weaker.

### 5.2. The second inclusion

Let $I = (xy, xz, yz) \subseteq \mathbb{C}[x, y, z]$ be the ideal of the union of the three coordinate axes. Using Howald's theorem and its asymptotic version one can compute all the ideals appearing in ($\star$). Since they are all integrally closed monomial ideals, we give them by giving their Newton polyhedra. Here $e = 2$; we take $k = 0$. First,

$$N_\bullet = \{(a, b, c) \mid\ a + b, a + c, b + c \geq 1\} \ni \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

We have $\operatorname{lct}(I) = \frac{3}{2}$ and $\operatorname{lct}(I^{(\bullet)}) = 2$, so we take $\ell = 1$. Now,

$$\begin{aligned}
\operatorname{Newt}[I^{(2r-1)}] &= \{(a, b, c) \mid\ a + b, a + c, b + c \geq 2r - 1,\ a + b + c \geq 3r - 1\},\\
\operatorname{Newt}[\mathcal{J}(2r \cdot I^{(\bullet)})] &= \{(a, b, c) \mid\ a + b, a + c, b + c \geq 2r - 1,\ a + b + c \geq 3r - 1\},\\
\operatorname{Newt}[(\mathcal{J}(2 \cdot I^{(\bullet)}))^r] &= \{(a, b, c) \mid\ a + b, a + c, b + c \geq r,\ a + b + c \geq 2r\},\\
\operatorname{Newt}[I^r] &= \{(a, b, c) \mid\ a + b, a + c, b + c \geq r,\ a + b + c \geq 2r\}.
\end{aligned}$$

This example shows that the place where improvements are needed is the second inclusion in ($\star$), which relies on the subadditivity theorem.

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# A combinatorial proof of non-speciality of systems with at most 9 imposed base points

**Abstract.** It is known that the Segre–Gimigliano–Harbourne–Hirschowitz Conjecture holds for linear systems of curves with at most 9 imposed base fat points. We give a nice proof based on a combinatorial method of showing non-speciality of such systems. We will also prove, by the same method, that systems $\mathcal{L}(km; m^{\times k^2})$ and $\mathcal{L}(km+1; m^{\times k^2})$ are non-special.

## 1. Introduction

Let $p_1, \ldots, p_r \in \mathbb{P}^2 = \mathbb{P}^2(\mathbb{K})$ be distinct points, where $\mathbb{K}$ is a field of characteristic 0. The points $p_1, \ldots, p_r$ will be called *imposed base points*. Let $m_1, \ldots, m_r$ be nonnegative integers. By $\mathcal{L}(d; m_1p_1, \ldots, m_rp_r)$ we denote the linear system of plane curves of degree $d$ with multiplicity at least $m_j$ at $p_j$, $j = 1, \ldots, r$. The dimension of $\mathcal{L}(d; m_1p_1, \ldots, m_rp_r)$ is upper semicontinuous in the position of imposed base points and reaches minimum for points in general position. This minimum will be denoted by

$$\dim \mathcal{L}(d; m_1, \ldots, m_r).$$

We will also write $\mathcal{L}(d; m_1, \ldots, m_r)$ for a system with imposed base points in general position, and $\mathcal{L}(d; m_1^{\times s_1}, \ldots, m_r^{\times s_r})$ for repeated multiplicities. Define the *virtual dimension of* $\mathcal{L}(d; m_1, \ldots, m_r)$

$$\operatorname{vdim} \mathcal{L}(d; m_1, \ldots, m_r) = \frac{d(d+3)}{2} - \sum_{j=1}^{r} \binom{m_j+1}{2}$$

and the *expected dimension of* $\mathcal{L}(d; m_1, \ldots, m_r)$

$$\operatorname{edim} \mathcal{L}(d; m_1, \ldots, m_r) = \max\{\operatorname{vdim} \mathcal{L}(d; m_1, \ldots, m_r), -1\}.$$

By linear algebra one has

$$\dim \mathcal{L}(d; m_1, \ldots, m_r) \geq \operatorname{edim} \mathcal{L}(d; m_1, \ldots, m_r)$$

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and $\mathcal{L}(d; m_1, \ldots, m_r)$ is said to be *special* if strict inequality holds for points in general position, *non-special* otherwise.

For systems $L = \mathcal{L}(d; m_1, \ldots, m_r)$, $L' = \mathcal{L}(d'; m'_1, \ldots, m'_r)$ we have the *intersection number* denoted by $L \cdot L'$,

$$L \cdot L' = dd' - \sum_{j=1}^{r} m_j m'_j.$$

Definition 1
The system $L = \mathcal{L}(d; m_1, \ldots, m_r)$ satisfying

- $\dim L = \operatorname{edim} L = 0$,
- *self-intersection* $L^2 = L \cdot L = -1$,
- the only curve in $L$ is irreducible,

will be called a $-1$-*system*.

A curve $C \subset \mathbb{P}^2$ is said to be *in the base locus of* $\mathcal{L}(d; m_1, \ldots, m_r)$ if $C$ is the component of each curve in $\mathcal{L}(d; m_1, \ldots, m_r)$. Observe that, by Bézout Theorem, if $L$ is nonempty and $L \cdot L' = -t < 0$ for $-1$-system $L'$, then the curve $C \in L'$ is in the base locus of $L$ at least $t$ times, i.e., the equation of each curve in $L$ is divisible by $f^t$, where $f$ is the equation of $C$. Such $C$ is said to be a *multiple* $-1$-*curve in the base locus*, and it forces the system to be special:

$$\dim L \overset{\text{(by Lemma 2)}}{=} \dim(L - tL') \geq \operatorname{vdim}(L - tL') \overset{\text{(by Lemma 2)}}{>} \operatorname{vdim} L,$$

thus, by nonemptiness of $L$, we have also

$$\dim L > \operatorname{edim} L.$$

Lemma 2
*Let* $L = \mathcal{L}(d; m_1, \ldots, m_r)$, *let* $L' = \mathcal{L}(d'; m'_1, \ldots, m'_r)$ *be a* $-1$-*system, let* $L - tL' = \mathcal{L}(d - d'; m_1 - m'_1, \ldots, m_r - m'_r)$. *If* $L \cdot L' = -t < 0$, *then*

$$\dim(L - tL') = \dim L,$$
$$\operatorname{vdim}(L - tL') = \operatorname{vdim} L + \frac{t^2 - t}{2}.$$

The proof of the Lemma is postponed to the next section. The system with multiple $-1$-curve in the base locus will be called $-1$-*special*. We have seen that every $-1$-special system is special. The following conjecture due to Harbourne [13], Gimigliano [10] and Hirschowitz [15] states the following.

Conjecture 3
*A system* $\mathcal{L}(d; m_1, \ldots, m_r)$ *with imposed base points in general position is special if and only if it is* $-1$-*special.*

In [5] it is shown that the above Conjecture is equivalent to the conjecture posed by Segre [18].

Conjecture 4
*If a system $L = \mathcal{L}(d; m_1, \ldots, m_r)$ with imposed base points in general position is special, then every curve in $L$ is non-reduced.*

We will refer to either one of the above conjectures as to Segre–Harbourne–Gimigliano–Hirschowitz (SHGH for short) Conjecture. From now on we will assume that imposed base points are always in general position.

The SHGH Conjecture can be reformulated using standard systems. A system $\mathcal{L}(d; m_1, \ldots, m_r)$ is called *standard* if $m_1 \geq m_2 \geq \ldots \geq m_r$ and

$$d \geq m_1 + m_2 + m_3.$$

Theorem 5
*In order to show that the SHGH Conjecture holds for at most $r$ points it suffices to show that each standard system for at most $r$ points is non-special.*

For completeness, we will give a proof of this well-known Theorem in the next section.

The fact that the SHGH Conjecture holds for $r \leq 9$ points has been shown by various methods in [16], [10] and [12], but the first results appeared already in [2]. A nice idea is to use the following well-known fact.

Proposition 6
*Let $d$, $m_1$, $m_2$, $m_3$ be nonnegative integers. If $d \geq m_1 + m_2 + m_3$, $m_1 \geq m_2 \geq m_3$ and the system $\mathcal{L}(d; m_1, m_2^{\times 3}, m_3^{\times 5})$ is non-special, then any standard system $\mathcal{L}(d; m_1, m_2, m_3, m_4, \ldots, m_9)$ is non-special.*

For completeness, we will give a proof of this proposition in the next section.

In the paper we will prove that SHGH holds for $r \leq 9$ points using only elementary facts based on linear algebra. In fact we must prove the following.

Theorem 7
*Let $d$, $m_1$, $m_2$, $m_3$ be nonnegative integers. If $d \geq m_1 + m_2 + m_3$ and $m_1 \geq m_2 \geq m_3$, then the system $\mathcal{L}(d; m_1, m_2^{\times 3}, m_3^{\times 5})$ is non-special.*

One of the main ingredients is the *cutting diagram algorithm* from [7]. Briefly, it is proved that in order to show non-speciality of a given system it suffices to find an appropriate finite set of points in $\mathbb{N}^2$ enjoying some combinatorial properties. To be precise, we must first define, for any finite $D \subset \mathbb{N}^2$, the system

$$\mathcal{L}(D; m_1, \ldots, m_r)$$

of polynomials with support in $D$ and with multiplicity at least $m_j$ at $p_j$, $j = 1, \ldots, r$. Formally, we identify $\mathbb{N}^2$ with monomials in $\mathbb{K}[X, Y]$

$$\mathbb{N}^2 \ni (x, y) \mapsto X^x Y^y \in \mathbb{K}[X, Y]$$

and put

$$\mathcal{L}(d; m_1, \ldots, m_r) = \{f \in \mathbb{K}[X, Y] : \ \mathrm{supp}(f) \in D, \ \mathrm{mult}_{p_j}(f) \geq m_j, \ j = 1, \ldots, k\}.$$

The set $\mathcal{L}(D; m_1, \dots, m_r)$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}[X, Y]$. We say that *conditions in $\mathcal{L}(D; m_1, \dots, m_r)$ are independent* if

$$\dim_{\mathbb{K}} \mathcal{L}(D; m_1, \dots, m_r) = \#D - \sum_{j=1}^{r} \binom{m_j + 1}{2}.$$

The system $\mathcal{L}(D; m_1, \dots, m_r)$ is called *empty* if

$$\dim_{\mathbb{K}} \mathcal{L}(D; m_1, \dots, m_r) = 0.$$

Observe that, by dehomogenizing and generality assumption, if conditions in $\mathcal{L}(D; m_1, \dots, m_r)$ are independent for $D = \{(x, y) : x + y \leq d\}$, then $\mathcal{L}(d; m_1, \dots, m_r)$ is non-special, similarly $\mathcal{L}(D; m_1, \dots, m_r)$ is empty if and only if $\mathcal{L}(d; m_1, \dots, m_r)$ is empty.

The cutting diagram algorithm is based on the following two theorems.

Theorem 8 ([7], Theorem 14)
*Let $D, D' \subset \mathbb{N}^2$ be finite, let $m_1, \dots, m_r, m'_1, \dots, m'_s$ be nonnegative integers. If*

- *$D \cap D' = \varnothing$,*
- *conditions in $\mathcal{L}(D; m_1, \dots, m_r)$ are independent (resp. $\mathcal{L}(D; m_1, \dots, m_r)$ is empty),*
- *conditions in $\mathcal{L}(D'; m'_1, \dots, m'_s)$ are independent (resp. $\mathcal{L}(D'; m'_1, \dots, m'_s)$ is empty),*
- *there exists an affine function $\mathbb{N}^2 \colon f \ni (a, b) \mapsto q_1 a + q_2 b + q_3 \in \mathbb{Q}$, $q_1, q_2, q_3 \in \mathbb{Q}$ such that $f$ has strictly negative values on $D$ and nonnegative values on $D'$,*

*then conditions in*

$$L = \mathcal{L}(D \cup D'; m_1, \dots, m_r, m'_1, \dots, m'_s)$$

*are independent (resp. $L$ is empty).*

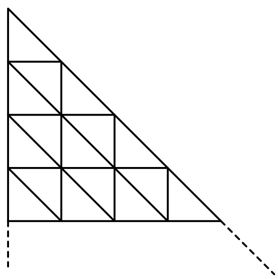
Theorem 9 ([7], Proposition 13)
*Let $D \subset \mathbb{N}^2$ be finite, let $m_1$ be a nonnegative integer. Then conditions in $\mathcal{L}(D; m_1)$ are independent if and only if $D$, considered as a set of points in $\mathbb{Q}^2$, does not lie on a curve of degree $m_1 - 1$. If $\#D = \binom{m_j+1}{2}$ and conditions in $\mathcal{L}(D; m_1)$ are independent, then $\mathcal{L}(D; m_1)$ is empty.*

The proofs are technical but use only simple linear algebra.

Theorem 10
*Let $k$, $m$ be nonnegative integers. Then systems $\mathcal{L}(km; m^{\times k^2})$ and $\mathcal{L}(km+1; m^{\times k^2})$ are non-special.*

It is known that the above theorem holds. More generally, homogeneous systems with the square number of imposed base points are always non-special, see [8]. Such systems, i.e., homogeneous with the number of imposed base points satisfying some property have been widely studied. For example, systems of the form $\mathcal{L}(d; m^{\times 4^h})$ have been considered in [9]; this consideration has been extended to systems of the form $\mathcal{L}(d; m^{\times 4^h 9^k})$ in [1]; systems with the number of imposed base points being nearly a square have been considered in [4]; systems of the form $\mathcal{L}(d; m_1^{\times 9}, m_2, \ldots, m_r)$ for $m_1 \geq m_2 \geq \ldots \geq m_r$ (so called *quasiuniform*) in [14], and systems of the form $\mathcal{L}(d; m^{\times r})$ for $r \geq 4m^2$ in [17].

The proof of Theorem 10 using toric degenerations can be found in [3]. We will give a simple combinatorial proof in a sequence of lemmas. Both proofs exploit the natural dissection of a two-dimensional simplex into $k^2$ simplexes:



but the idea behind is slightly different. In the degeneration approach one controls the behaviour of the system "along" the intersection of two meeting regions (given always by weak inequalities). In our approach it is better to completely separate regions by defining them with strict inequalities.

Lemma 11
*Conditions in the system $\mathcal{L}(D; m^{\times 16})$ are independent for*

$$D = \{(x, y) \in \mathbb{N}^2 : \ x + y \leq 4m + 1\};$$

*conditions in the system $\mathcal{L}(D; m^{\times 25})$ are independent for*

$$D = \{(x, y) \in \mathbb{N}^2 : \ x + y \leq 5m + 1\};$$

*thus systems $\mathcal{L}(4m + 1; m^{\times 16})$ and $\mathcal{L}(5m + 1; m^{\times 25})$ are non-special.*

Lemma 12
*Systems $\mathcal{L}(4m; m^{\times 16})$, $\mathcal{L}(5m; m^{\times 25})$, $\mathcal{L}(6m; m^{\times 36})$ and $\mathcal{L}(6m+1; m^{\times 36})$ are empty.*

Lemma 13
*Systems $\mathcal{L}(km; m^{\times k^2})$ and $\mathcal{L}(km + 1; m^{\times k^2})$ are empty for $k \geq 7$.*

Proofs of lemmas are postponed to the next section.

## 2. Proofs

*Proof of Lemma 2.* To prove that $\dim(L - tL') = \dim L$ observe that multiplication by the equation of $C \in L'$ in $t$th power induces an isomorphism between $L - tL'$ and $L$. By a straightforward calculation one shows that

$$\operatorname{vdim}(L - tL') = \operatorname{vdim} L - tL \cdot L' + \frac{t^2 L'^2}{2} + \frac{t(-3d' + \sum_{j=1}^{r} m'_j)}{2}.$$

Moreover,

$$L'^2 - 2\operatorname{vdim} L' = -3d' + \sum_{j=1}^{r} m'_j,$$

which completes the proof.

*Proof of Theorem 5.* Let $L = \mathcal{L}(d; m_1, \ldots, m_r)$. Consider the following procedure:

Step 1. Sort multiplicities in non-increasing order.

Step 2. If $k = d - m_1 - m_2 < 0$, then take $d \longleftarrow d + k$, $m_1 \longleftarrow m_1 + k$, $m_2 \longleftarrow m_2 + k$ and go back to Step 1.

Step 3. If $k = d - m_1 - m_2 - m_3 < 0$, then take $d \longleftarrow d + k$, $m_j \longleftarrow m_j + k$ for $j = 1, 2, 3$ and go back to Step 1.

We finish with a system $L'$. We will show that in each step the dimension does not change. Indeed, if $k = d - m_1 - m_2$ is negative, then each curve in $\mathcal{L}(d; m_1, m_2, m_3, \ldots)$ is reducible and contains the line passing through $p_1$, $p_2$ at least $-k$ times. In other words, we have the isomorphism

$$\varphi\colon \mathcal{L}(d - k; m_1 - k, m_2 - k, m_3, \ldots) \to \mathcal{L}(d; m_1, m_2, m_3, \ldots)$$

given by multiplication by the equation of the line in $k$th power. In Step 3 the result follows from applying the Cremona transformation based on $p_1$, $p_2$, $p_3$ to our system (see eg. [11, Section 3]). This transformation induces the isomorphism

$$\varphi\colon \mathcal{L}(d - k; m_1 - k, m_2 - k, m_3 - k, m_4, \ldots) \to \mathcal{L}(d; m_1, m_2, m_3, m_4, \ldots)$$

(the proof of this fact using only linear algebra can be found in [6, proof of Theorem 3]; we use the fact that the system passed Step 2, so $d - m_1 - m_2 \geq 0$). By an easy computation one can show that the virtual dimension does not change in Step 3, while in Step 2 it increases by $\frac{k^2+k}{2}$. Thus for $k \leq -2$ we obtain $L$ to be either empty or special. In the second case, we know that after some Cremona transformations there exists a multiple line in the base locus. Again, by easy computations we can show that Cremona transformation preserves the intersection number, hence the multiple line from the base locus will be mapped, by the reversed process, into a multiple $-1$-curve in the base locus of $L$. Therefore $L$ is either $-1$-special or enjoys the same properties (dimension, virtual dimension, emptiness, speciality...) as $L'$, which is standard.

*Proof of Proposition 6.* Assume, by hypothesis, that $L_2 = \mathcal{L}(d; m_1, \ldots, m_9)$ is special. We will show that $L_1 = \mathcal{L}(d; m_1, m_2^{\times 3}, m_3^{\times 5})$ is special. Let $c$ be the difference between the number of conditions in $L_1$ and the number of conditions in $L_2$,

$$c = \binom{m_1+1}{2} + 3\binom{m_2+1}{2} + 5\binom{m_3+1}{2} - \sum_{j=1}^{9} \binom{m_j+1}{2}.$$

Since each condition can lower the dimension by at most one, we have

$$\dim L_1 \geq \dim L_2 - c > \operatorname{edim} L_2 - c \geq \operatorname{vdim} L_2 - c = \operatorname{vdim} L_1.$$

Since for $d \geq m_1 + m_2 + m_3$, the virtual dimension

$$\begin{aligned} \operatorname{vdim} L_1 &\geq \frac{(m_1+m_2+m_3)(m_1+m_2+m_3+3)}{2} \\ &\quad - \frac{m_1(m_1+1) + 3m_2(m_2+1) + 5m_3(m_3+1)}{2} \\ &= (m_1 - m_3) + m_2(m_1 - m_2) + m_3(m_1 + m_2 - 2m_3) \\ &\geq 0, \end{aligned}$$

we have $\operatorname{vdim} L_1 = \operatorname{edim} L_1$ and consequently

$$\dim L_1 > \operatorname{edim} L_1.$$

Before proving Theorem 7 we must prepare some helpful systems with independent conditions.

Definition 14

Let $m$ be a positive integer. Define an *$m$-rectangle* to be the set

$$\left\{(x,y) \in \mathbb{N}^2 : \ a - \frac{1}{2} < x < a + m + \frac{1}{2}, \ b - \frac{1}{2} < y < b + m - \frac{1}{2}\right\}$$

or the set

$$\left\{(x,y) \in \mathbb{N}^2 : \ a - \frac{1}{2} < x < a + m - \frac{1}{2}, \ b - \frac{1}{2} < y < b + m + \frac{1}{2}\right\}$$

for some nonnegative integers $a$, $b$. Define an *$m$-triangle* to be the set

$$\left\{(x,y) \in \mathbb{N}^2 : \ x > a - \frac{1}{2}, \ y > a - \frac{1}{2}, \ x + y < 2a + m - \frac{1}{2}\right\}$$

for some nonnegative integer $a$. The examples are shown on Figure 1.

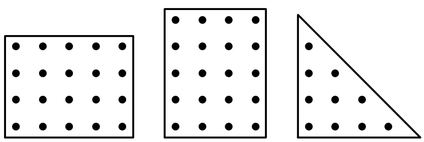


**Figure 1.** Example of 4-rectangles and 4-triangle

Lemma 15

*Let $T$ be an $m$-triangle, let $R$ be an $m$-rectangle. Then conditions in the systems $\mathcal{L}(T; m)$ and $\mathcal{L}(R; m^{\times 2})$ are independent and these systems are empty.*

*Proof.* Observe that there exists parallel lines $\ell_1, \dots, \ell_m$ such that $\#(T \cap \ell_j) = j$. The proof for $\mathcal{L}(T; m)$ is completed by Theorem 9 and Bézout Theorem.

To deal with $\mathcal{L}(R; m^{\times 2})$ observe that $R$ can be divided into two pieces $R_1$, $R_2$, such that $R_1$ is an $m$-triangle, while $R_2$ is a rotated $m$-triangle. By Theorem 8 the proof is completed.

*Proof of Theorem 7.* Let $D = \{(x, y) \in \mathbb{N}^2 : x+y \le d\}$. We want to show that conditions in $\mathcal{L}(D; m_1, m_2^{\times 3}, m_3^{\times 5})$ are independent. Take the following cutting of $D$ into three pieces:

$$D_1 = \Big\{(x, y) \in D : \ y > m_2 + m_3 + \frac{1}{2}\Big\},$$
$$D_2 = \Big\{(x, y) \in D : \ y < m_2 + m_3 + \frac{1}{2} \ \text{ and } \ (m_3 + 2)y + x > m_3^2 + 3m_3 - \frac{1}{2}\Big\},$$
$$D_3 = \Big\{(x, y) \in D : \ (m_3 + 2)y + x < m_3^2 + 3m_3 - \frac{1}{2}\Big\}.$$

By Theorem 8 it is enough to show that conditions in systems $\mathcal{L}(D_1; m_1)$, $\mathcal{L}(D_2; m_2^{\times 3})$, $\mathcal{L}(D_3; m_3^{\times 5})$ are independent. Observe that, by easy computations, an $m_1$-triangle with vertices $(0, m_2 + m_3 + 1)$, $(m_1 - 1, m_2 + m_3 + 1)$ and $(0, m_1 + m_2 + m_3)$ is contained in $D_1$. Similarly, observe that an $m_2$-rectangle with vertices $(0, m_3 + 1)$, $(m_2, m_3 + 1)$, $(m_2, m_3 + m_2)$, $(0, m_3 + m_2)$ and an $m_2$-triangle with vertices $(m_2 + 1, m_3)$, $(2m_2, m_3)$, $(m_2 + 1, m_3 + m_2 - 1)$ are contained in $D_2$. Moreover, these two shapes can be separated from each other by an affine line. For $D_3$, we take three shapes — an $m_3$-rectangle with vertices $(0, 0)$, $(m_3 - 1, 0)$, $(m_3 - 1, m_3)$, $(0, m_3)$, another $m_3$-rectangle with vertices $(m_3, 0)$, $(2m_3, 0)$, $(2m_3, m_3 - 1)$, $(m_3, m_3 - 1)$ and finally an $m_3$-triangle with vertices $(2m_3 + 1, 0)$, $(3m_3, 0)$, $(2m_3 + 1, m_3 - 1)$. By Theorem 8 and Lemma 15 the proof is completed.

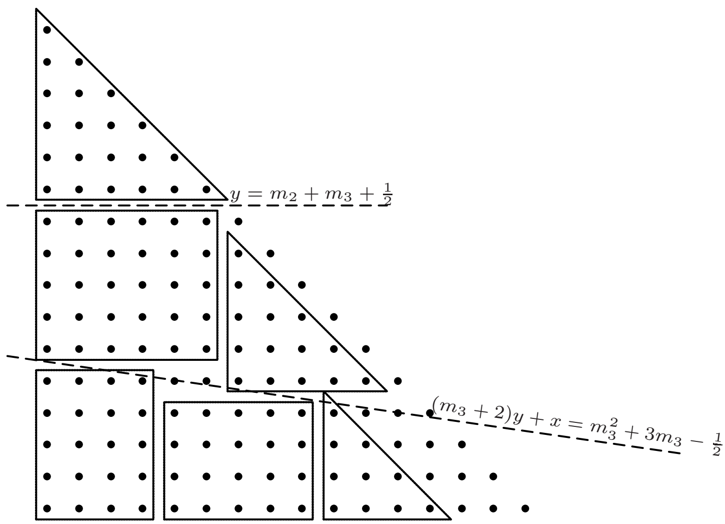



**Figure 2.** Example of divisions for $m_1 = 6$, $m_2 = 5$, $m_3 = 4$

*Proof of Lemma 11.* The proofs can be easily read off from Figures 3 and 4. The pictures are drawn for $m = 3$, but can be easily rescaled. Less obvious cuttings are presented, the details are left to the reader. By $\varepsilon$ we denote a sufficiently small positive rational number.

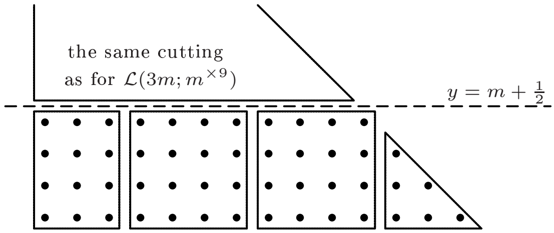



**Figure 3.** Divisions for $\mathcal{L}(4m + 1; m^{\times 16})$

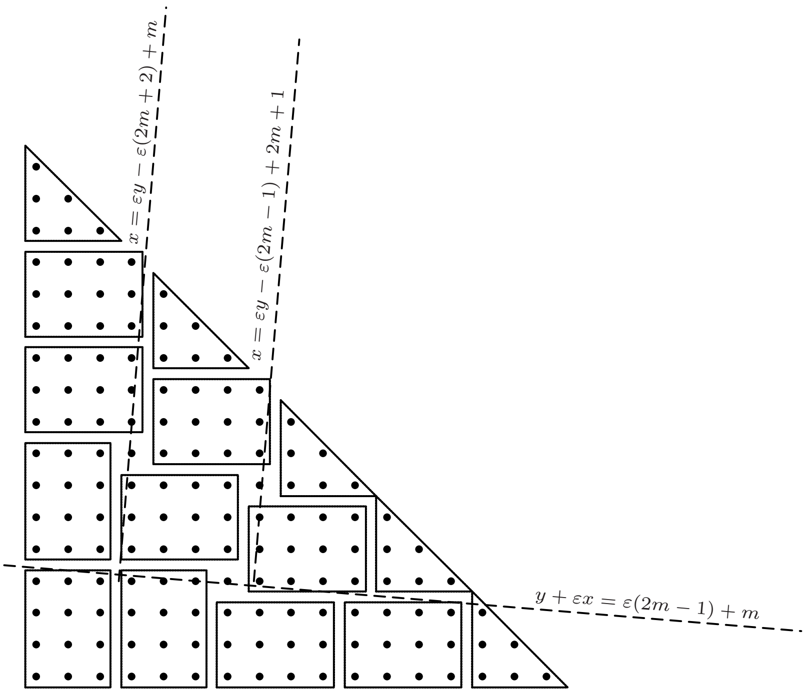



**Figure 4.** Divisions for $\mathcal{L}(5m + 1; m^{\times 25})$

*Proof of Lemma 12.* Emptiness of $\mathcal{L}(6m; m^{\times 36})$ would follow from emptiness of $\mathcal{L}(6m+1; m^{\times 36})$. Again, the proofs can be easily read off from Figures 5, 6 and 7. Observe that if $R \subset \mathbb{N}^2$ is contained in some $m$-rectangle, then $\mathcal{L}(R; m^{\times 2})$ is empty.

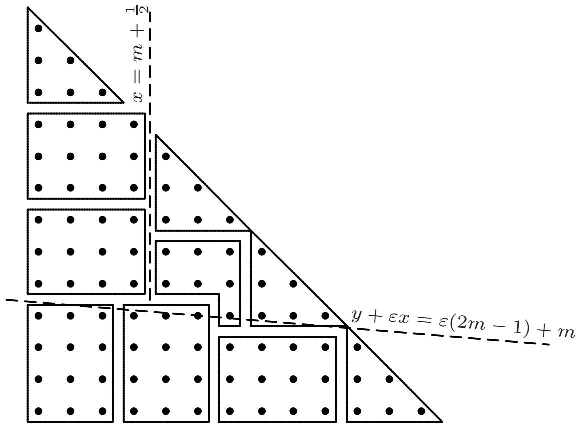



**Figure 5.** Divisions for $\mathcal{L}(4m; m^{\times 16})$

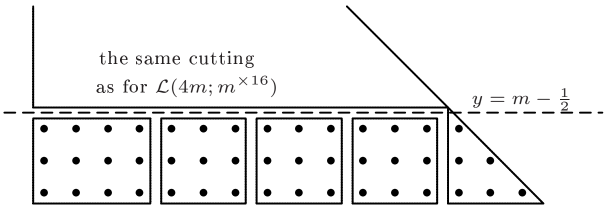



**Figure 6.** Divisions for $\mathcal{L}(5m; m^{\times 25})$

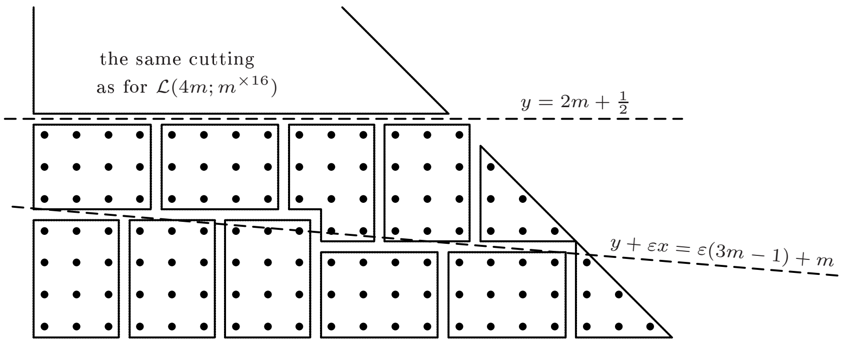



**Figure 7.** Divisions for $\mathcal{L}(6m + 1; m^{\times 36})$

*Proof of Lemma 13.* Emptiness of $\mathcal{L}(km; m^{\times k^2})$ would follow from emptiness of $\mathcal{L}(km + 1; m^{\times k^2})$. The first cutting, into upper and bottom part, is given by the line $y = m - \frac{1}{2}$. Since $k - 1 \geq 6$, we use induction to the upper part, cutting it exactly as $\mathcal{L}((k - 1)m + 1; m^{\times (k-1)^2})$. The bottom part

$$B = \{(x, y) \in \mathbb{N}^2 : \ x + y \leq km + 1, \ y \leq m\}$$

gives the system $\mathcal{L}(B; m^{\times(2k-1)})$. We will cover $B$ from right to left with one $m$-triangle and $(k-1)$ $m$-rectangles of hight $m$. This allows to cover $(k-1)(m+1)+m = km+k-1$ lattice points $(x,0) \in B$, while $\#\{(x,0) \in B\} = km+2$. Thus we can entirely cover $B$ and the proof is completed.

Remark 16
There is no theoretical obstruction to make similar proofs for systems of the form $\mathcal{L}(km+k_0; m^{\times k^2})$ for fixed $k_0$. In fact, for $k$ satisfying $k \geq k_0+2$ the induction step (emptiness of $\mathcal{L}(km+k_0; m^{\times k^2})$ implies emptiness of $\mathcal{L}((k+1)m+k_0; m^{\times (k+1)^2})$) will work. One can even hope that for $k$'s satisfying $k \leq K+1$,

$$K = \max\{k : \ \operatorname{vdim} \mathcal{L}(km+k_0; m^{\times k^2}) \geq 0 \text{ for some } m\},$$

it is always possible to prove non-speciality by the presented method.

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# Kinetic equation for a gas with attractive forces as a functional equation

**Abstract.** Diffusion problems studied in the time scale comparable with time of particles collision lead to kinetic equations which for step-wise potentials are functional equations in the velocity space. After a survey of derivation of kinetic equations by projective operator method, an attention is paid to the Lorentz gas with step potential. The gas is composed of $N$ particles: $N-1$ of which are immovable; between those $N-1$ immovable particles – scatterers, particle number 1 is moving, and we describe its movement by means of one-particle distribution function satisfying a kinetic equation. Solutions of the kinetic equation for some simple potentials are given. We derive also a kinetic equation for one-dimensional Lorentz gas, which is a functional equation.

## 1. Introduction

General kinetic equations with convolution time integral (hence nonlocal in time and non-markovian) were first derived and discussed by the Brussels group, headed by Ilya Prigogine, [1]. Different correlation functions used to describe non-equilibrium processes satisfy such equations, [2] – [7].

A comparison of the theory of the Brussels group, with the Bogolyubov theory, then being developed by the Uhlenbeck group was given in a paper by Stecki and Taylor, [8]. These results were next extended and ordered by the Brussels group, [9].

Robert Zwanzig, [4, 10] described a new method of derivation of kinetic equations. The main tool of this derivation is the use of projection operators in the Hilbert space of Gibbsian ensemble densities. It was noted by Nelkin and Ghatak that the Van Hove self-correlation function $G_s(r,t)$ for a dilute fluid is determined by a linearized Boltzmann equation identical to that occuring in the theory of neutron diffusion, [11].

The kinetic equation (KE) describing diffusion in time scale comparable with time of the particles collision, is also a time convolution kinetic equation, which for

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a step-wise interaction potential takes form of a functional equation in the velocity space.

We work in the framework of kinetic theory of a Boltzmann gas, with use of statistical mechanics methods. The gas is composed of $N$ particles, and the problem discussed concerns the diffusion of a marked particle (number 1) amid $(N-1)$ other classical dilute gas particles.

Applying to the Liouville equation the proper projection operator, a kinetic equation for one-particle distribution function $f(\mathbf{k}, \mathbf{v}_1, t)$ is derived. Here $\mathbf{k}$ denotes the Fourier vector variable (wave vector) after transformation of spacial coordinate $\mathbf{r}_1$, which denotes the position of particle number 1. The vector $\mathbf{v}_1$ is the velocity of this particle, while $t$ is a time. Function $f(\mathbf{k}, \mathbf{v}_1, t)$ is Fourier transform of one-particle distribution function $f_s(\mathbf{r}, \mathbf{v}_1, t)$, which represents the probability of finding a particle at time $t$ at $\mathbf{r}$ with velocity $\mathbf{v}_1$, if the same particle was at time $t = 0$ at $\mathbf{r} = \mathbf{0}$ with the given distribution of velocity $\mathbf{v}_1$, e.g. the Maxwellian.

Right-hand side of KE has a form of time convolution of a scattering operator $\mathcal{G} = \mathcal{G}(\mathbf{k}, t)$ and function $f = f(\mathbf{k}, \mathbf{v}_1, t)$. It is valid not only for long times (in comparison with time of collision, as it is in case of the Boltzmann equation and in Brownian movement theory) but also for short times.

KE considered here was found previously by Jan Stecki, [12], cf. also [13, 14]. This is a time convolution equation for a gas which particles interact by attractive-repelling potential with step dependence on distance. In such a case the phase space consists of distincly separated regions and the kinetic equations is transformed from a convolutive one into a functional equation.

### 1.1. Notation

The gas occupies volume $V$ and consists of $N$ particles, numbered by indices $i = 1, \ldots, N$, and $m_i$, $\mathbf{v}_i$ and $\mathbf{r}_i$ are the mass, velocity and position of particle number $i$, respectively. Cartesian coordinates of vector $\mathbf{v}_i$ are denoted by $v_{ix}, v_{iy}, v_{iz}$ and those of $\mathbf{r}_i$ by $x_i, y_i, z_i$.

The Maxwell distribution function of the velocity is denoted by

$$\varphi_M(v_i) = \sqrt{\Big(\beta\frac{m}{2\pi}\Big)^3} \exp\Big(-\beta m\frac{v_i^2}{2}\Big).$$

Here the velocity modulus $v_i = |\mathbf{v}_i|$ is used and $v_i^2 = v_{ix}^2 + v_{iy}^2 + v_{iz}^2$, while $\beta^{-1} = k_\mathrm{B}T$ with the Boltzmann constant $k_\mathrm{B}$ and absolute temperature $T$.

The temperature of an ideal gas is related to its average kinetic energy *per* particle by the relation

$$\bar{E}_\mathrm{kin} = \frac{3}{2}k_\mathrm{B}T = \frac{3}{2\beta}.$$

The second law of thermodynamics states that any two interacting systems will reach the same average energy *per* particle and hence the same temperature. In equilibrium, the probability of finding a particle with velocity $\mathbf{v}_i$ in the infinitesimal element $d\mathbf{v}_i = [dv_{ix}, dv_{iy}, dv_{iz}]$ about velocity $\mathbf{v}_i = [v_{ix}, v_{iy}, v_{iz}]$ is $\varphi_M(v_i)dv_{ix}dv_{iy}dv_{iz}$ or $\varphi_M(v_i)d\mathbf{v}_i$.

The interaction potential $u_{ij}$ between particles number $i$ and number $j$ depends on distance between these particles only:

$$u_{ij} = u_{ij}(|\mathbf{r}_i - \mathbf{r}_j|).$$

Hence the total potential energy of the system

$$U = \sum_{i<j}^{N} u(|r_i - r_j|) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} u_{ij}(r_{ij}) = \sum_{i<j} u_{ij},$$

where $r_{ij} = |\mathbf{r}_{ij}| = |\mathbf{r}_i - \mathbf{r}_j|$.

## 1.2. Physical meaning

The function $f = f(\mathbf{k}, \mathbf{v}_1, t)$ is related to scattering phenomena. Essential for interpretation of incoherent scattering experiments is the Van Hove function

$$G_s(|\mathbf{r} - \mathbf{r}_0|, t) = \Big\langle \frac{V}{N} \sum_{i=1}^{N} \delta(\mathbf{r}_i(0) - \mathbf{r}_0)\delta(\mathbf{r}_i(t) - \mathbf{r}) \Big\rangle \tag{1}$$

where

$$\langle(\ldots)\rangle = \int \frac{1}{Z_N} \mathrm{e}^{-\beta H} (\ldots)\, d\mathbf{v}^N d\mathbf{r}^N \quad \text{with} \quad Z_N = \int \mathrm{e}^{-\beta H}\, d\mathbf{v}^N d\mathbf{r}^N$$

denotes the canonical average.

The function $G_s(\mathbf{r}, t)$ represents the probability of finding a particle at $\mathbf{r}$ at time $t$ if the same particle was at $\mathbf{r} = 0$ at time $t = 0$.

Van Hove law for incoherent scattering reads

$$S_s(\mathbf{k}, \omega) = \frac{1}{2\pi} \int \exp[i(\mathbf{kr} - \omega t)] G_s(\mathbf{r}, t)\, d\mathbf{r} dt = \frac{1}{2\pi} \int \exp(-i\omega t) I_s(\mathbf{k}, t)\, dt,$$

where

$$I_s(\mathbf{k}, t) = \int \exp(i\mathbf{kr}) G_s(\mathbf{r}, t)\, d\mathbf{r}$$

cf. [15] – [19]. On the other hand, we have

$$I_s(\mathbf{k}, t) = \int f(\mathbf{k}, \mathbf{v}_1, t)\, d\mathbf{v}_1$$

and $I_s(\mathbf{k}, t)$ is the Fourier transform of $G_s(\mathbf{r}, t)$ and function $f(\mathbf{k}, \mathbf{v}_1, t)$ can be found by kinetic theory. Namely, it satisfies the following linear KE

$$\Big(\frac{\partial}{\partial t} + i\mathbf{k}\mathbf{v}_1\Big) f(\mathbf{k}, \mathbf{v}_1, t) = \int_0^t \mathcal{G}(\mathbf{k}, \tau) f(\mathbf{k}, \mathbf{v}_1, t - \tau)\, d\tau \tag{2}$$

where

$$f(\mathbf{k}, \mathbf{v}_1, t) = \int d\mathbf{r}_1\, \mathrm{e}^{-i\mathbf{k}\mathbf{r}_1} \int d\mathbf{v}^{N-1}\, F_N(t)$$

and

$$F_N(t) = \mathrm{e}^{-tK_N} F_N(0)$$

with

$$F_N(0) = \mathrm{e}^{i\mathbf{k}\mathbf{r}_1} \varphi_M(v_1) \cdot \ldots \cdot \varphi_M(v_N) \frac{\mathrm{e}^{-\beta U}}{Q} \tag{3}$$

Here $U = \sum_{i<j} u_{ij}$ and

$$K_N = \sum_{i=1}^{N} v_i \frac{\partial}{\partial \mathbf{r}_i} - \sum_{i<j} \frac{\partial U}{\partial \mathbf{r}_i} \frac{1}{m_i} \frac{\partial}{\partial \mathbf{v}_i} \tag{4}$$

is the $N$-particle Liouville operator.

Normalization factor in (3)

$$Q = \int_V \mathrm{e}^{-\beta U}\, d\mathbf{r}^N, \quad \text{where } d\mathbf{r}^N = d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_N$$

is known as the partition function or sum-over-states.

The partition function $Q$ is related to thermodynamical properties of the system, cf. [20], [21], [22]. With a model of the microscopic constituents of a system, one can calculate the microstate energies, and thus the partition function, which will then allow us to calculate all the other thermodynamical properties of the system.

Research in the prediction of binding affinities has been a continuing effort for more than half a century, [23, 24]. An important application of the configuration integral lies in the development of computational models for the ligand-receptor binding affinities. Their study constitutes the most important problem in computational biochemistry. Especialy, the prediction of absolute ligand-receptor binding affinities is essential in a wide range of biophysical questions, from the study of protein-protein interactions to structure-based drug design.

In a ligand-receptor binding, a ligand is in general any molecule that binds to another molecule; the receiving molecule is called a receptor, which is a protein on the cell membrane or within the cell cytoplasm. Such binding can be represented by the chemical reaction describing noncovalent molecular association $A + B \leftrightarrow AB$, where $A$ represents the protein (receptor), $B$ the ligand molecule, and $AB$ the protein-ligand complex. The change in the Gibbs free energy can be expressed as a ratio of configuration integrals, [25].

An alternative form of the kinetic equation (2) is

$$(-iz + i\mathbf{k}\mathbf{v}_1) f(\mathbf{k}, \mathbf{v}_1, z) - f(\mathbf{k}, \mathbf{v}_1, t = 0) = G(\mathbf{k}, z) f(\mathbf{k}, \mathbf{v}_1, z) \tag{5}$$

where $f(\mathbf{k}, \mathbf{v}_1, z)$ is the Laplace transform of $f(\mathbf{k}, \mathbf{v}_1, t)$ defined as $f(z) = \int_0^\infty \mathrm{e}^{izt} f(t)\, dt$. We use the same letter for a function and its Laplace transform, but it does not lead to confusion, because all arguments are explicitly written.

If $m_1 \gg m_i$, $i = 2, 3, \ldots, N$ we have the Brownian diffusion of particle number 1. If $m_1 \ll m_i$, $i = 2, 3, \ldots, N$ - the Lorentz gas is dealt with, cf. also [26, 27].

## 1.3. Diffusion in biology

For big times and for isotropic medium the Van Hove function $G_s = G_s(r,t)$ is given by a solution of the classical Fick's equation, namely,

$$G_s(r,t)_{t\to\infty} = \frac{1}{8(\pi D t)^{\frac{3}{2}}} \mathrm{e}^{\frac{-r^2}{4Dt}},$$

where $D$ denotes the (macroscopic) diffusion coefficient. After tranformations we get $I_s(k,t)_{t\to\infty} = \exp(-k^2 D t)$ and

$$S_s(k,\omega) = \frac{1}{\pi}\frac{Dk^2}{\omega^2 + (Dk^2)^2}.$$

Hence

$$D = \lim_{\omega\to 0}\lim_{k\to 0} \pi \frac{\omega^2 S_s(k,\omega)}{k^2}.$$

We have also

$$D = -\lim_{t\to\infty} \frac{1}{6t}\frac{\partial^2 I_s(k,t)}{\partial k^2}.$$

In spite of passing to the limit, residual information about the dynamics of system is still contained in the diffusion coefficient $D$. For example, in the random walk diffusion the coefficient $D = \frac{h^2}{2\tau}$, with $h$ and $\tau$ being the length and duration of one step in the walk, respectively.

The laws of diffusion (in which coefficient $D$ is used) were discovered in 1855 by physician and physiologist Adolf Eugen Fick, [28] – [30].

At the beginning of the 20th century, Einstein and Smoluchowski, independently, have found relation between macroscopic diffusion coefficient $D$ and the Brownian movement phenomenon, explaining it in microscopic, molecular terms, cf. [31, 32]. The phenomenon was first explicitly described in 1828 by the physician and botanist Robert Brown, who observed in aqueous suspensions of pollen grains from *Clarkia pulchella* a rapid, continuous, short-range motion of small included particles that "arose neither from currents in the fluid nor from its gradual evaporation, but belonged to the particle itself", [33, 34].

After discovery of Fick's laws, in physiology dominated the opinion that diffusion laws should explain all problems of metabolism. It was widely believed in XIX century that diffusion is responsible for such organic processes as gas exchange in the leaves of plants, gas exchange in the lungs of animals, the uptake of the products of digestion from the gut.

However, the development of knowledge on the cell structure has permitted to gather an abundant evidence on inadequacy of diffusion theory for explaining much of the movements of substances in organisms, studied in biology and medicine. The Fick diffusion alone could described physiological processes only in dead tissues.

In 1912 medical doctor and physiologist, Otto Heinrich Warburg published a discovery: oxygen utilization requires structural elements in the cell – a solid phase. These structures, now recognized as mitochondria, had been described by light microscopists two decades before Warburg's publication, and 80 years later were found to be places where Brownian motors work, [35].

The assumptions of the Einstein–Smoluchowski model are not even approximately met *in vivo*. The cell contains a highly concentrated and heterogeneous assembly of deformable, interacting and inelastically colliding particles; much of the solvent (water) is bound to solid structures which, although not necessarily long-living, have huge surface areas; and in any case the conditions only tend to thermodynamic equilibrium after death. The model representing the "microscopic" aspect of diffusion theory assumes a dilute, homogeneous suspension of rigid, non-interacting and elastically colliding particles, a monophasic system with the solvent (largely) unbound, and a tendency towards equilibrium. Also, the model assumes that there are no net solvent movements, and this is undoubtedly relevant in intracellular transport, [36, 37], also [38].

After the idea arose that the cell internum does, at least in part, behave as a gel, the diffusion through gels became an important subject of study. Investigations of diffusion in gels put a question on applicability of Fick's laws in the field.

Bigwood has shown in 1930 that not only is diffusion in gels highly dependent on the absolute concentration of diffusing substance (in contrast to the classical linear Fick's theory that diffusion rates depend only on concentration gradients), but that it is both slow and unpredictable, particularly when the gel is made of protein, as the gel state of the cell internum should be, cf. [39, 40]. It became clear then that in description of biological cell extreme order has to be reconciled with a fluid anatomy. Two kinds of intracellular transport are possible: one, which accounts for the movements of macromolecules and assemblies; and second, which will account for the movements of small molecules and ions, [41].

In 1949 Hans Ussing conducted investigations with use of radioactive tracers and gave the systematic molecular level account of a "secretion" process in biology, as an opposite to the "diffusion" description. Ussing defined the term "active transport", which means the creation of a genuinely "uphill" concentration gradient, cf. [42, 43]. Active transport is now an accepted part of biological knowledge, and individual active transport mechanisms are frequently objects of research.

In 1950 BBC lecture J.Z. Young concluded: the more we come to know of the flux of chemical changes in the body, the more one great weakness of the machine analogy stands out. The concept of a dynamic organization, such as that of a whirlpool, demands a consideration of time – of before and after and of gradual development and change of pattern, but the machine models of physiology allow no place for this element. In the tissue spaces, as well as inside the cell, there is fluid circulation among solid-state elements, [44].

The diffusion concepts persisted for a long time in description of respiratory processes. Until now, the method of "diffusion capacity" is practicised as a measurement of the lungs ability to transfer gases. Oxygen absorption may be limited by diffusion in circumstances of low ambient oxygen or high pulmonary blood flow. Carbon dioxide is not limited by diffusion under most circumstances. The "diffusion capacity" is part of comprehensive test series of lung function called pulmonary function testing. It is known, however, diffusivity estimates are seriously problematic even with modern equipment. Longmuir wrote: "If simple diffusion is the sole mechanism of tissue oxygen transport as proposed by Krogh (1919), it is difficult to see how acclimatization could occur without a reduction in the diffusion coefficient. The kinetics of oxygen transport cannot be explained by passive diffusion alone; the search for other mechanisms led to the observation that all kinetic data could be explained by channels in cells along which the oxygen diffuses faster than in water, [45, 46]."

The cell internum is far more complex organised right down to the molecular level than was hitherto appreciated, to the point where ideas of a relatively solid-state chemistry model have occoured. The flow theory of enzyme kinetics — a role of solid geometry in the control reaction velocity in live animals. This contrasts sharply with the former concept that diffusion is the way by which molecules interact within an aqueous solution of the cell internum, [47] – [52].

In living systems, most molecules do not generally move, but are moved, when we consider what would happen if everything depended upon Brownian motion and the law of mass action. R.P.C. Johnson in 1983 recognised a grey area at the molecular level when considering the movement of molecules within living cells: "This is the region of scale where flow and diffusion are not clearly separated; where the concepts of temperature and molecular movement overlap; where it is not clear whether molecules move or are moved; where the ideas of active and passive lose their meaning", [53, 48], also [54] and [55].

Until now, biologists use the term "diffusion" in a twofold meaning. One is Fick's diffusion, and the second one is vernacular, for spreading process, when "diffusion" is not adhered to a specific, defined scientific term. For an active transport the term active diffusion is sometimes used, as an opposite to passive (i.e. Fickian) diffusion.

The complication in the description of biological processes may be found in application of the Smoluchowski diffusion with drift equation. In this equation an aleatory aspect is coupled with deterministic. The drift force controls diffusion and diffusion reflects the influence of thermal vibrations of the evironment on the process.

All phenomena, biological also, are developing in given thermal conditions, and the application of thermodynamics is inevitable. The "microscopic" aspect of diffusion theory, is that random thermal motions of molecules in liquids are responsible for return of diffusion, particularly Brownian movement theories, into contemporary biophysics.

Brownian or molecular motors are biological "nanomachinees" and are the essential agents of movement in living organisms. A motor is regarded as a device that consumes energy and converts it into motion or mechanical power. Adenosine triphosphate (ATP) is the fuel for the molecular motors action. Many protein-based molecular motors convert the chemical energy present in ATP into mechanical energy. The ATPase molecular motors are found in the membranes of mitochondria, the microscopic bodies in the cells of nearly all living organisms, as well as in chloroplasts of plant cells, where the enzyme is responsible for converting food to usable energy, [56] and [57].

It was shown by Streater that the Smoluchowski equation for a Brownian particle potentially can be supplemented by an equation for the dynamics of the temperature, so that the first and the second laws of thermodynamics are obeyed. He considered also a model studied by David Smith, known as the dumbbell model, in which the Brownian particle is a two-level atom, and had shown that under isothermal conditions, the free energy can be given a natural definition out of equilibrium, and is a decreasing function of time, [58], also [59]. Smith has applied his model to describe a myosin molecule, [60, 61], also [62] and [63].

Macromolecular particles playing a role in protein motors are heavy (Brownian) in comparison with solvent (water) molecules, but are light (Lorentzian) in comparison with mass of substratum (mitochondrium).

Another biological example in which the passive diffusion plays a role is provided by alimentation processes in cartilage, tissue which supplies smooth surfaces for the movement of articulating bones. The cartilage is built of cells, called chondrocytes, producing a large amount of extracellular matrix composed of collagen fibers, abundant ground substance rich in proteoglycan, and elastin fibers. Unlike other connective tissues, cartilage does not contain blood vessels. The chondrocytes are fed by diffusion, helped by the pumping action generated by compression of the articular cartilage or flexion of the elastic cartilage. Thus, compared to other connective tissues, cartilage grows and repairs more slowly, [64].

The diffusion process appears in biology also as the property of homeostasis in organisms.

Homeostasis (from Greek: hómos, "equal"; and istemi, "to stand" lit. "to stand equally"; coined by Walter Bradford Cannon) is the property of either an open system or a closed system, especially a living organism, that regulates its internal environment so as to maintain a stable, constant condition. Multiple dynamic equilibrium adjustment and regulation mechanisms make homeostasis possible. The concept came from that of milieu interieur that was created by Claude Bernard, often considered as the father of physiology, and published in 1865.

With respect to any given life system parameter, an organism may be a conformer or a regulator. Regulators try to maintain the parameter at a constant level over possibly wide ambient environmental variations. On the other hand, conformers allow the environment to determine the parameter. For instance, endothermic animals maintain a constant body temperature, while exothermic animals exhibit wide body temperature variation. Examples of endothermic animals include mammals and birds, examples of exothermic animals include reptiles and some sea animals.

Most homeostatic regulation is controlled by the release of hormones into the bloodstream. However other regulatory processes rely on simple diffusion to maintain a balance.

Homeostatic regulation extends far beyond the control of temperature. All animals also regulate their blood glucose, as well as the concentration of their blood. Mammals regulate their blood glucose with insulin and glucagon. These hormones are released by the pancreas, the inadequate production of the two for any reason, would result in diabetes. The kidneys are used to remove excess water and ions from the blood. These are then expelled as urine. The kidneys perform a vital role in homeostatic regulation in mammals, removing excess water, salt, and urea from the blood. These are the body's main waste products, [65].

## 2. Projective operator method

The projection operator is introduced, [66],

$$\mathcal{P} = \mathrm{e}^{i\mathbf{k}\mathbf{r}_1} \frac{f_N^0}{\varphi_M(v_1)} \int d\mathbf{v}^{N-1} d\mathbf{r}^N \, \mathrm{e}^{-i\mathbf{k}\mathbf{r}_1},$$

where

$$f_N^0 = \prod_{i=1}^{N} \varphi_M(v_i) \frac{1}{Q} \mathrm{e}^{-\beta U}$$

is the equilibrium distribution function. We observe

$$\mathcal{P} F_N(t) = \mathrm{e}^{i\mathbf{k}\mathbf{r}_1} \frac{f_N^0}{\varphi_M(v_1)} f(\mathbf{k}, \mathbf{v}_1, t).$$

In particular

$$\mathcal{P} F_N(0) = \mathrm{e}^{i\mathbf{k}\mathbf{r}_1} \frac{f_N^0}{\varphi_M(v_1)} f(\mathbf{k}, \mathbf{v}_1, 0) = \mathrm{e}^{i\mathbf{k}\mathbf{r}_1} f_N^0 = F_N(0)$$

and

$$(1 - \mathcal{P}) F_N(0) = 0.$$

Also

$$\int d\mathbf{v}^{N-1} d\mathbf{r}^N \, \mathrm{e}^{-i\mathbf{k}\mathbf{r}_1} \mathcal{P} F_N(t) = f(\mathbf{k}, \mathbf{v}_1, t).$$

The Liouville equation

$$\frac{\partial}{\partial t} F_N(t) = -K_N F_N(t)$$

with $K_N$ given by (4), is now rewritten in the form

$$\frac{\partial}{\partial t} [\mathcal{P} F_N(t)] = -\mathcal{P} K_N \mathcal{P} F_N(t) - \mathcal{P} K_N (1 - \mathcal{P}) F_N(t)$$

and

$$\frac{\partial}{\partial t} [(1 - \mathcal{P}) F_N(t)] = -(1 - \mathcal{P}) K_N \mathcal{P} F_N(t) - (1 - \mathcal{P}) K_N (1 - \mathcal{P}) F_N(t).$$

Hence

$$\frac{\partial}{\partial t} [\mathcal{P} F_N(t)] = -\mathcal{P} K_N \mathcal{P} F_N(t) + \mathcal{P} K_N \int_0^t \mathrm{e}^{-\tau(1-\mathcal{P})K_N} (1 - \mathcal{P}) K_N \mathcal{P} F_N(t - \tau) \, d\tau$$

and finally

$$\Big(\frac{\partial}{\partial t}+i\mathbf{k}\mathbf{v}_1\Big)f(\mathbf{k},\mathbf{v}_1,t)$$
$$=\int d\mathbf{v}^{N-1}d\mathbf{r}^N\,\mathrm{e}^{-i\mathbf{k}\mathbf{r}_1}\mathcal{P}K_N\int\limits_0^t \mathrm{e}^{-\tau(1-\mathcal{P})K_N}\mathrm{e}^{i\mathbf{k}\mathbf{r}_1}\frac{f_N^0}{\varphi_M(v_1)}f(\mathbf{k},\mathbf{v}_1,t-\tau)\,d\tau$$

it is a general form of KE, correct also for small times, compared to the time of collision.

## 3. Density expansion

An alternative form of the kinetic equation (2) is

$$(-iz+i\mathbf{k}\mathbf{v}_1)f(k,v_1,z)=G(k,z)f(\mathbf{k},\mathbf{v}_1,z)+f(\mathbf{k},\mathbf{v}_1,t=0),$$

where

$$f(t=0)=\varphi_M(v_1)=\Big(\frac{2\pi}{\beta m}\Big)^{-\frac{3}{2}}\mathrm{e}^{-\frac{1}{2}\beta v_1^2}$$

with $f(\mathbf{k},\mathbf{v}_1,z)$ being Laplace transform of $f(\mathbf{k},\mathbf{v}_1,t)$

$$f(z)=\int\limits_0^\infty \mathrm{e}^{izt}f(t)\,dt.$$

The scattering operator in (2)

$$G(\tau)=\int d\mathbf{r}^N d\mathbf{v}^{N-1}\,\mathrm{e}^{-i\mathbf{k}\mathbf{r}_1}K_N\mathrm{e}^{-\tau(1-\mathcal{P})K_N}(1-\mathcal{P})K_N\mathrm{e}^{i\mathbf{k}\mathbf{r}_1}f_N^0\frac{1}{\varphi_M(v_1)}.$$

After Laplace transformation we get the equation

$$(-iz+i\mathbf{k}\mathbf{v}_1)f(\mathbf{k},\mathbf{v}_1,z)-\varphi_M(v_1)$$
$$=\int d\mathbf{v}^{N-1}d\mathbf{r}^N\,\mathrm{e}^{-i\mathbf{k}\mathbf{r}_1}\mathcal{P}K_N\frac{1}{-iz+(1-\mathcal{P})K_N}$$
$$\times(1-\mathcal{P})K_N\mathrm{e}^{i\mathbf{k}\mathbf{r}_1}\frac{f_N^0}{\varphi_M(v_1)}f(\mathbf{k},\mathbf{v}_1,z)$$

which right-hand side can be written as

$$G(\mathbf{k},z)f(\mathbf{k},\mathbf{v}_1,z)=\int d\mathbf{v}^{N-1}d\mathbf{r}^N\,\mathrm{e}^{-i\mathbf{k}\mathbf{r}_1}K_N\frac{1}{-iz}\frac{1}{1-\frac{1}{iz}(1-\mathcal{P})K_N}$$
$$\times(1-\mathcal{P})\,K_N\mathrm{e}^{i\mathbf{k}\mathbf{r}_1}\frac{f_N^0}{\varphi_M(v_1)}f(\mathbf{k},\mathbf{v}_1,z).$$

The first terms of the expansion are

$$G(\mathbf{k},z) = \int d\mathbf{v}^{N-1} d\mathbf{r}^N \, \mathrm{e}^{-i\mathbf{k}\mathbf{r}_1} \Big[\frac{1}{iz}(K_N K_N - K_N \mathcal{P} K_N)$$
$$+ \Big(\frac{1}{iz}\Big)^2 (K_N K_N K_N - K_N K_N \mathcal{P} K_N - K_N \mathcal{P} K_N K_N + K_N \mathcal{P} K_N \mathcal{P} K_N)$$
$$+ \Big(\frac{1}{iz}\Big)^3 (\ldots) + \ldots \Big] \mathrm{e}^{i\mathbf{k}\mathbf{r}_1} \frac{f_N^0}{\varphi_M(v_1)}.$$

In the dilute gas approximation only linear terms with respect to $\rho = \frac{N}{V}$ are kept, and the following form of binary scattering operator is obtained

$$G_{12}(\mathbf{k},z) = \frac{N-1}{V^2} \iiint d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{v}_2 \, (-iz + i\mathbf{k}\mathbf{v}_1)\mathrm{e}^{-i\mathbf{k}\mathbf{r}_1}$$
$$\times \int_0^\infty dt \, \mathrm{e}^{izt} \big(\mathrm{e}^{-tK_2} - \mathrm{e}^{-tK_2^0}\big)(-iz + i\mathbf{k}\mathbf{v}_1)\mathrm{e}^{i\mathbf{k}\mathbf{r}_1} \mathrm{e}^{-\beta u} \varphi_M(v_2).$$

For $\mathbf{k} = \mathbf{0}$ and $z = 0$ the scattering operator reduces to the Boltzmann scattering operator. It also takes the Boltzmann form for $\mathbf{k} = \mathbf{0}$, arbitrary $z$ and sufficiently high velocity $v_1$.

## 4. Lorentz gas

The Lorentz gas corresponds to the case $m_2 \to \infty$, $v_2 \to 0$ and $\varphi_M(v_2) \to \delta(v_2)$. Only the velocity of particle 1 remains and is denoted by $\mathbf{v}_1 = \mathbf{v}$. The Lorentz model is widely studied as a simple model of a crystal, cf. for example [67] – [78].

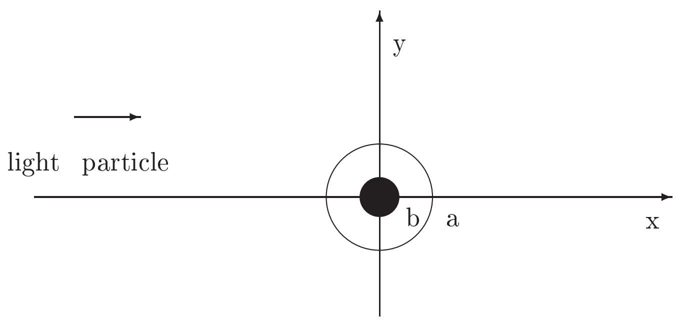


**Figure 1.** Spherical potential: hard core of radius $b$ (black circle) and well (white ring) with internal radius $b$ and external radius $a$

The Lorentz gas was examined in [66] for the following case of repulsive – attractive potential, see Figure 1,

$$u(r) = \infty \quad \text{if } r < b, \qquad u(r) = -u_0 < 0 \quad \text{if } b < r < a, \qquad u(r) = 0 \quad \text{if } r > a,$$

where $r$ is the radius in polar coordinates. Thus, the potential possesses spherical rigid repulsive core of radius $b$ surrounded by a well ($b < r < a$) of depth $-u_0$,

$u_0 > 0$. Scattering operator for this potential, for the dilute Lorentz gas has the following form

$$G_{12}f(\mathbf{k},z,\mathbf{v}) = i(-z+\mathbf{kv})\frac{N}{V}\int d\mathbf{r}\,\mathrm{e}^{-\beta u}\varphi_M(v)\mathrm{e}^{-i\mathbf{kr}}$$
$$\times\int_{t_1}^{\infty} dt\,\mathrm{e}^{izt}\big(\mathrm{e}^{-tK_2}-\mathrm{e}^{-tK_2^0}\big)\mathrm{e}^{i\mathbf{kr}}i(-z+\mathbf{kv})\frac{f(\mathbf{k},z,\mathbf{v})}{\varphi_M(v)}.$$

The KE for three-dimensional Lorentz gas of $N-1$ fixed rigid spheres with the square-well attractive potential was given also in [66]. It is an integral (in configurational space) and functional (in velocity space) equation for the unknown distribution function $\psi(v)$ which links the values of $\psi(v)$ at 8 different values of argument $v$.

## 5. Lorentz gas of rigid spheres with finite time of collision $\tau^*$

The potential of rigid sphere with rectangular well changes the time of interaction of the light particle with scatterer, is contrast to the zero time of interaction with the rigid sphere potential alone. To avoid additional consideration of scattering trajectory we accept the rigid sphere potential ($R_1 = R_2$), in which, however, the interacting particles remain connected for a certain time $\tau^*$. This time of collision is negative in case of the potential well. In this case

$$G_{12}f(\mathbf{k},z,\mathbf{v}) = v\varphi_M(v)\frac{N}{V}\frac{a^2}{4}\int d\Omega\,\big[\Psi(\mathbf{k},z,\mathbf{v}')\mathrm{e}^{iz\tau^*}-\Psi(\mathbf{k},z,\mathbf{v})+1-\mathrm{e}^{iz\tau^*}\big],$$

where integration is performed over the full solid angle and

$$\Psi(\mathbf{k},z,\mathbf{v}) \equiv \frac{f(\mathbf{k},z,\mathbf{v})}{\varphi_M(v)}.$$

We introduce the following notation

$$\pi a^2 v\frac{N}{V} = \varepsilon_0^{-1}, \qquad \frac{1}{4\pi}\int d\Omega = \hat{P}.$$

Kinetic equation takes the form

$$(-iz+i\mathbf{kv}+\varepsilon_0^{-1})\Psi - h = \varepsilon_0^{-1}\mathrm{e}^{iz\tau^*}(\hat{P}\Psi)+1-\mathrm{e}^{iz\tau^*}.$$

Here $h = \delta(v-v')$ is the initial condition. Hence

$$\Psi = \frac{\varepsilon_0^{-1}\mathrm{e}^{iz\tau^*}}{-iz+i\mathbf{kv}+\varepsilon_0^{-1}}\hat{P}\Psi + \frac{h+1-\mathrm{e}^{iz\tau^*}}{-iz+i\mathbf{kv}+\varepsilon_0^{-1}}.$$

Therefore the solution reads

$$\Psi = \frac{\varepsilon_0^{-1}\mathrm{e}^{iz\tau^*}}{-iz+i\mathbf{kv}+\varepsilon_0^{-1}}\left(1-\frac{\mathrm{e}^{iz\tau^*}}{\mathrm{kv}\varepsilon_0}\arctan\frac{\mathrm{kv}\varepsilon_0}{1-i\varepsilon_0 z}\right)^{-1}$$
$$\times\hat{P}\frac{h+1-\mathrm{e}^{iz\tau^*}}{-iz+i\mathbf{kv}+\varepsilon_0^{-1}} + \frac{h+1-\mathrm{e}^{iz\tau^*}}{-iz+i\mathbf{kv}+\varepsilon_0^{-1}}.$$

For the hydrodynamic pole we have

$$-iz = \varepsilon_0^{-1} + \mathbf{kv}\cot[(\cos z\tau^* - i\sin z\tau^*)\mathbf{kv}\varepsilon_0].$$

If the time of collision $\tau^* = 0$, KE equation becomes

$$(-iz + i\mathbf{kv} + \varepsilon_0^{-1})\Psi - h = v\varphi_M(v)\frac{N}{V}\frac{a^2}{4}\int d\Omega.$$

This is the classical Boltzmann equation for the Lorentz gas. Its solution has the form discussed by Hauge in [78].

## 6. One-dimensional KE

The 3 dimensional dynamics, even for the Lorentz gas, is still too complicated to be effectively solved and for this reason we limit ourselves to 1-dimensional model. It posseses some important features of 3-dimensional case, but mechanics of the light particle motion is more simple. It may be expected that the obtained results will have a more general meaning. Such procedure is often used, see [79] – [82].

The one-dimensional considerations permitted Fermi, Pasta, Ulam and Mary Tsingou to find that the behaviour of a 32-atom chain is quite different from intuitive expectation. Instead of thermalisation, a complicated quasi-periodic behaviour of the system was observed, [83], also [84].

Morita and Fukui considered the heat transfer in one-dimensional gas, [85], while Kac [86] – [89] and McKean [90] considered one-dimensional analogues of the linear Boltzmann equation.

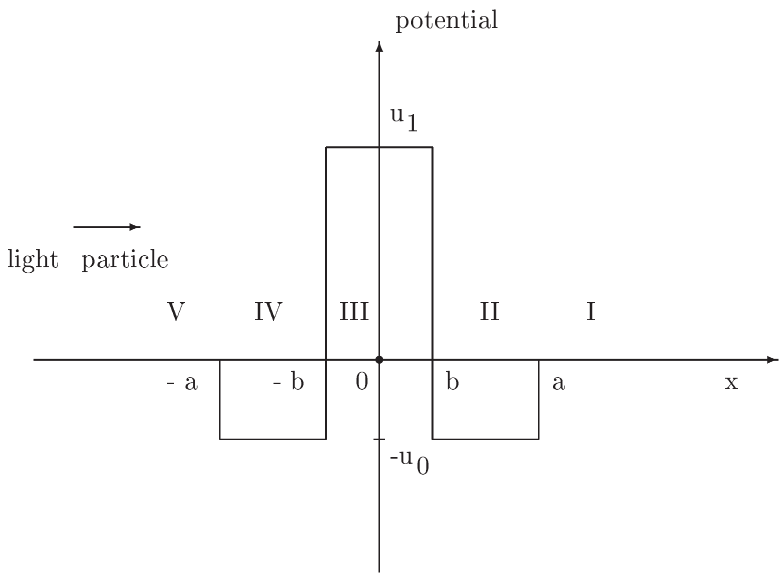


**Figure 2.** Configurational space of one-dimensional model. Light particle moves in potential of a well of depth $-u_0 < 0$ and a repulsive core of hight $u_1 > 0$

The Lorentz gas is examined here in one dimension, for the case of attractive – repulsive potential

$$u(x) = \begin{cases} u_1, & |x| < b, \\ -u_0 < 0, & b < |x| < a, \\ 0, & |x| > a. \end{cases}$$

The quantity $-u_0$, with $u_0 > 0$ is the depth of the potential well, while $u_1 > 0$ denotes the height of the potential barrier, see Figure 2.

## 6.1. Kinetic equation in 1 dimension

The KE has still structure of (5) but vectors are now one-dimensional

$$(-iz + ikv)f(k, v, z) - f(k, v, t = 0) = G(k, z)f(k, v, z).$$

Scattering operator for the dilute Lorentz gas of $N$ particles in one-dimensional segment $L$, $(-\frac{L}{2} < x < \frac{L}{2})$, has the following form

$$\begin{aligned} G(k, z)f(k, v, z) = i(-z + kv)\frac{N}{L}\int dx\, \mathrm{e}^{-\beta u}\varphi_M(v)\mathrm{e}^{-ikx} \\ \times \int_{t_1}^{\infty} dt\, \mathrm{e}^{izt}\big(\mathrm{e}^{-tK_2} - \mathrm{e}^{-tK_2^0}\big)\mathrm{e}^{ikx} i(-z + kv)\frac{f(k, z, v)}{\varphi_M(v)} \end{aligned} \tag{6}$$

Here $K_2$ is the two particle Liouville operator, see (4), for $N = 2$. In calculations $L \to \infty$ but $\frac{N}{L}$ is kept constant. Such procedure is known as the thermodynamic limit (one increases the volume together with the particle number so that the average particle number density remains constant). Thus, integration with respect to $x$ extends from minus to plus infinity. Below we put

$$\Psi(v) = \frac{f(k, z, v)}{\varphi_M(v)}.$$

The phase space is now two-dimensional only: one-dimension for positions and another for velocities of the light particle. The position space is divided into 5 regions, from I to V, see Figure 2, while the velocity space in each of these regions is divided, in dependence of kinetic energy of the particle (whether it permits for bounded or unbounded motion of the particle).

## 6.2. Bounded motions

The bounded motion of particle occurs in regions of the potential well, II and IV, only, if simultaneously the particle kinetic energy is less than the depth of the well $u_0$.

**Regions $b \leq x \leq a$ and $-a \leq x \leq -b$**

Let us consider bounded motion of our particle in segment $b \leq x \leq a$ with velocity $v < \sqrt{\frac{2}{m}u_0}$. The position of particle along its trajectory is given by relation

$$
\begin{aligned}
\mathrm{e}^{-tK_2}x &= x(-t) \\
&= x - vt\eta(t_1 - t) - [vt_1 + v'(t - t_1)]\eta(t - t_1)\eta(t_2 - t) \\
&\quad - [vt_1 + v'(t_2 - t_1) + v''(t - t_2)]\eta(t - t_2)\eta(t_3 - t) \\
&\quad - [vt_1 + v'(t_2 - t_1) + v''(t_3 - t_2) + v'''(t - t_3)]\eta(t - \tau_3)\eta(t_4 - t) \\
&\quad - \ldots - [vt_1 + v'(t_2 - t_1) + v''(t_3 - t_2) + v'''(t_4 - t_3) + v''''(t_5 - t_4) \\
&\quad + \ldots + v^{(2n)}(t_{2n+1} - t_{2n}) + v^{(2n+1)}(t - t_{2n+1})]\eta(t - \tau_{2n+1}).
\end{aligned}
$$

Similarly, the velocity is given by

$$
\begin{aligned}
\mathrm{e}^{-tK_2}v &= v(-t) \\
&= v\eta(t_1 - t) + v'(t - t_1)\eta(t_2 - t) + v'(t - t_1)\eta(t_2 - t) \\
&\quad + v''\eta(t - t_2)\eta(t_3 - t) + v'''\eta(t - t_3)\eta(t_4 - t) \\
&\quad + \ldots + v^{(2n-1)}\eta(t - t_{2n-1})\eta(t_{2n} - t) \\
&\quad + v^{(2n)}\eta(t - t_{2n})\eta(t_{2n+1} - t) + v^{(2n+1)}\eta(t - t_{2n+1}).
\end{aligned}
$$

In the equation above we have

$$
v' = -v, \quad v'' = v, \ldots, v^{(2n-1)} = -v, \quad v^{(2n)} = v
$$

and $2n$ denotes the number of full periods performed by the particle in the time $t$. Moreover, $t_m$, $m = 1, 2, \ldots$ denotes the moment of bouncing from the wall of the well. The instant of the first collision of the particle with wall is given by

$$
t_1 = \frac{x - b}{|v|} \tag{7}
$$

and the next instants satisfy relations

$$
t_2 - t_1 = t_3 - t_2 = \ldots = t_m - t_{m-1} = \tau = \frac{a - b}{|v|}.
$$

Differences between the subsequent moments are identical and equal $\tau$. Therefore the period of bouncing is $2\tau$.

For the time being we replace the infinity in the upper limit of time integral in (6) by $T$, and next extend $T \to \infty$ and $n \to \infty$.

$$
\begin{aligned}
&(G(k,z)f(k,v,z))_{IIA} \\
&= i(-z + kv)\frac{N}{L}\mathrm{e}^{\beta u_0}\varphi_M(v)\int_b^a dx\,\mathrm{e}^{-ikx}\left\{\int_{t_1}^{t_2} dt\,\mathrm{e}^{i(z+kv)t}\mathrm{e}^{-ik2vt_1}i(-z - kv)t\Psi(-v)\right.
\end{aligned}
$$

$$+\int_{t_2}^{t_3} dt\, \mathrm{e}^{i(z-kv)t}\mathrm{e}^{-ik[vt_1-v(t_2-t_1)-vt_2]}i(-z+kv)t\Psi(v)$$

$$+\int_{t_3}^{t_4} dt\, \mathrm{e}^{i(z+kv)t}\mathrm{e}^{-ik[vt_1+vt_3]}i(-z-kv)t\Psi(-v)$$

$$+\int_{t_4}^{t_5} dt\, \mathrm{e}^{i(z-kv)t}\mathrm{e}^{-ik[vt_1-v(t_4-t_3)-vt_4]}i(-z+kv)t\Psi(v)$$

$$+\ldots+\int_{t_{2n-1}}^{t_{2n}} dt\, \mathrm{e}^{i(z+kv)t}\mathrm{e}^{-ik[vt_1+vt_{2n-1}]}i(-z-kv)t\Psi(-v)$$

$$+\int_{t_{2n}}^{t_{2n+1}} dt\, \mathrm{e}^{i(z-kv)t}\mathrm{e}^{-ik[vt_1-v(t_{2n}-t_{2n-1})-vt_{2n}]}i(-z+kv)t\Psi(v)$$

$$+\int_{t_{2n+1}}^{T} dt\, \mathrm{e}^{i(z+kv)t}\mathrm{e}^{-ik[vt_1+vt_{2n+1}]}i(-z-kv)t\Psi(-v)$$

$$\left.-\int_{t_1}^{T} dt\, \mathrm{e}^{i(z-kv)t}i(-z+kv)t\Psi(v)\right\}.$$

We take $n$ so large that

$$T-t_{2n+1}<\tau.$$

We integrate at first with respect to $t$, and next with respect to $x$. Variable $x$ is found only in time of the first collision $t_1=\frac{x-b}{v}$, cf. (7). After integration and passing with $n$ to infinity, there appear series ot type

$$1+\mathrm{e}^{iz2\tau}+\mathrm{e}^{iz4\tau}+\ldots+\mathrm{e}^{iz2n\tau}+\ldots=\frac{1}{1-\mathrm{e}^{iz2\tau}} \qquad \text{for } n\to\infty.$$

Finally we find the following KE

$$(-iz+ikv)\Psi(v)-h(v)=C[\Psi(-v)-\Psi(v)]$$

with

$$h(v)=\frac{f(k,v,t=0)}{\varphi_M(v)} \quad \text{and} \quad C=\frac{N}{L}|v|\frac{1-2\mathrm{e}^{iz\tau}\cos(kv\tau)+\mathrm{e}^{iz\tau}}{1-\mathrm{e}^{iz\tau}}\mathrm{e}^{\beta u_0}.$$

Remark that $C$ is even in $v$. The solution of KE reads

$$\Psi(v)=\frac{(-iz-ikv+C)h(v)+Ch(-v)}{-z^2-2izC+k^2v^2}.$$

Identical relation describes the bound motion in segment $-a\le x\le -b$, with velocity $v<\sqrt{\frac{2}{m}u_0}$.

### 6.3. Unbounded motions

The phase subspaces of bounded and unbounded one-dimensional motions of the particle are separated by the value of its kinetic energy, in the dilute gas approximation. The particle once trapped in bounded motion, persists in it forever, and a particle in the phase subspace where unbounded motion occurs can never become bounded.

#### 6.3.1. Region I: ($a < x < \infty$)

The particle which is at the time $t = 0$ in this region is subject to 3 accelerations if its kinetic energy is less then the height of the potential barrier $u_1$ (Case IA) or 4 accelerations if it is higher (Case IB).

**Case IA:** if $0 < v < \sqrt{\frac{2}{m}u_1}$ we have

$$\begin{aligned} x(-t) = x - vt\eta(t_1 - t) &- [vt_1 + v'(t - t_1)]\eta(t - t_1)\eta(t_2 - t) \\ &- [vt_1 + v'(t_2 - t_1) + v''(t - t_2)]\eta(t - t_2)\eta(t_3 - t) \\ &- [vt_1 + v'(t_2 - t_1) + v''(t_3 - t_2) + v'''(t - t_3)]\eta(t - t_3) \end{aligned}$$

and

$$v(-t) = v\eta(t_1 - t) + v'\eta(t_2 - t)\eta(t - t_1) + v''\eta(t_3 - t)\eta(t - t_2) + v'''\eta(t - t_3)$$

with

$$v' = \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}u_0}, \quad v'' = -v', \quad v''' = -v \tag{8}$$

and

$$t_1 = \frac{x - a}{|v|}, \quad t_2 = t_1 + \frac{a - b}{|v'|}, \quad t_3 = t_2 + \frac{a - b}{|v'|} = t_1 + 2\frac{a - b}{|v'|}$$

denote the moments of subsequent collisions. As before (Section 6.2), the position variable $x$ is hidden in $t_1$.

After straightforward calculations we get the part of right hand side of (6) linked to this subregion

$$\begin{aligned} Gf_{(IA)} = \frac{N}{L}|v|\varphi_M(v)\Big\{&\Big[1 - \mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\Big]\Psi(v') \\ &+ \Big[1 - \mathrm{e}^{i(z+kv')\frac{a-b}{|v'|}}\Big]\mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\Psi(-v') + \mathrm{e}^{iz2\frac{a-b}{|v'|}}\Psi(-v) - \Psi(v)\Big\} \end{aligned}$$

**Case IB:** if $v > \sqrt{\frac{2}{m}u_1}$ we have

$$\begin{aligned} x(-t) = x - vt\eta(t_1 - t) &- [vt_1 + v'(t - t_1)]\eta(t - t_1)\eta(t_2 - t) \\ &- [vt_1 + v'(t_2 - t_1) + v''(t - t_2)]\eta(t - t_2)\eta(t_3 - t) \\ &- [vt_1 + v'(t_2 - t_1) + v''(t_3 - t_2) + v'''(t - t_3)]\eta(t - \tau_3)\eta(t_4 - t) \\ &- [vt_1 + v'(t_2 - t_1) + v''(t_3 - t_2) + v'''(t_4 - t_3) + v''''(t - t_4)]\eta(t - t_4) \end{aligned}$$

and

$$v(-t) = v\eta(t_1 - t) + v'\eta(t_2 - t)\eta(t - t_1) + v''\eta(t_3 - t)\eta(t - t_2) \\ + v'''\eta(t_4 - t)\eta(t - t_3) + v''''\eta(t - t_4)$$

with

$$v' = \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}u_0}, \quad v'' = \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}u_1}, \quad v''' = v', \quad v'''' = v \tag{9}$$

and

$$t_1 = \frac{x-a}{v}, \quad t_2 = t_1 + \frac{a-b}{v'}, \quad t_3 = t_2 + \frac{2b}{v''} \quad t_4 = t_3 + \frac{a-b}{v'}.$$

In this subregion

$$Gf_{(IB)} = \frac{N}{L}|v|\varphi_M(v)\Big\{\Big[1 - \mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\Big]\Big[1 + \mathrm{e}^{i(z-kv'')\frac{2b}{|v''|}}\mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\Big]\Psi(v') \\ + \Big[1 - \mathrm{e}^{i(z+kv'')\frac{2b}{|v''|}}\Big]\mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\Psi(v'') \\ - \Big[1 - \mathrm{e}^{i(z-kv')2\frac{a-b}{|v'|}}\mathrm{e}^{i(z-kv'')\frac{2b}{|v''|}}\Big]\Psi(v)\Big\}.$$

### 6.3.2. Region II: $b < x < a$

The bounded motion in this region was described in Section 6.2.

The particle which is at the time $t = 0$ in this region and has kinetic energy higher than the depth of the well $u_0$, is in an unbounded motion and has undergone 2 accelerations if its kinetic energy is lower than the hight of potential barrier $u_1$ (Case IIA) or 3 accelerations if its kinetic energy is higher than the barrier (Case IIB).

**Case IIA:** if $\sqrt{\frac{2}{m}u_1} > v > \sqrt{\frac{2}{m}u_0}$ we have

$$x(-t) = x - vt\eta(t_1 - t) - [vt_1 + v'(t - t_1)]\eta(t - t_1)\eta(t_2 - t) \\ - [vt_1 + v'(t_2 - t_1) + v''(t - t_2)]\eta(t - t_2)$$

and

$$v(-t) = v\eta(t_1 - t) + v'\eta(t_2 - t)\eta(t - t_1) + v''\eta(t - t_2)$$

with

$$v' = -v, \quad v'' = -\frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}u_0} \qquad \text{and} \qquad t_1 = \frac{x-b}{|v|}, \quad t_2 = t_1 + \frac{a-b}{|v|}.$$

Now

$$Gf_{(IIA)} = \frac{N}{L}|v|\mathrm{e}^{\beta u_0}\varphi_M(v)\Big[1 - \mathrm{e}^{-i(z+kv)\frac{a-b}{|v|}}\Big] \\ \Big\{\Big[1 - \mathrm{e}^{i(z+kv)\frac{a-b}{|v|}}\Big]\Psi(-v) + \mathrm{e}^{i(z+kv)\frac{a-b}{|v|}}\Psi(v'')) - \Psi(v)\Big\}.$$

**Case IIB:** if $v > \sqrt{\frac{2}{m}u_1}$ we have

$$\begin{aligned} x(-t) = {} & x - vt\eta(t_1 - t) - [vt_1 + v'(t - t_1)]\eta(t - t_1)\eta(t_2 - t) \\ & - [vt_1 + v'(t_2 - t_1) + v''(t - t_2)]\eta(t - t_2)\eta(t_3 - t) \\ & - [vt_1 + v'(t_2 - t_1) + v''(t_3 - t_2) + v'''(t - t_3)]\eta(t - \tau_3) \end{aligned}$$

and

$$v(-t) = v\eta(t_1 - t) + v'\eta(t_2 - t)\eta(t - t_1) + v''\eta(t - t_2)\eta(t_3 - t) + v'''\eta(t - t_3)$$

with

$$v' = \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}(u_0 + u_1)}, \quad v'' = v, \quad v''' = \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}u_0} \tag{10}$$

and

$$t_1 = \frac{x - b}{v}, \quad t_2 = t_1 + \frac{2b}{|v\prime|} \quad t_3 = t_2 + \frac{a - b}{|v|}.$$

Now

$$\begin{aligned} Gf_{(IIB)} = {} & \frac{N}{L}\,|v|\,\mathrm{e}^{\beta u_0}\varphi_M(v)\Big[1 - \mathrm{e}^{i(z-kv)\frac{a-b}{|v|}}\Big] \\ & \times \Big\{\Big[1 - \mathrm{e}^{i(z-kv')\frac{2b}{|v'|}}\Big]\Psi(v') + \mathrm{e}^{i(z-kv')\frac{2b}{|v'|}}\mathrm{e}^{i(z-kv)\frac{a-b}{|v|}}\,\Psi(v''') \\ & + \Big(\Big[1 - \mathrm{e}^{i(z-kv)\frac{a-b}{|v|}}\Big]\mathrm{e}^{i(z-kv')\frac{2b}{|v'|}} - 1\Big)\,\Psi(v)\Big\}. \end{aligned}$$

### 6.3.3. Region III: $-b < x < b$

The particle being at $t = 0$ in this region, has undergone 2 accelerations. The time dependence of its position and velocity is the following

$$\begin{aligned} x(-t) = {} & x - vt\eta(t_1 - t) - [vt_1 + v'(t - t_1)]\eta(t - t_1)\eta(t_2 - t) \\ & - [vt_1 + v'(t_2 - t_1) + v''(t - t_2)]\eta(t - t_2) \end{aligned}$$

and

$$v(-t) = v\eta(t_1 - t) + v'\eta(t_2 - t)\eta(t - t_1) + v''\eta(t - t_2)$$

with

$$v' = \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}(u_0 + u_1)}, \quad v'' = \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}u_1} \tag{11}$$

and

$$t_1 = \frac{x + b}{|v|}, \quad t_2 = t_1 + \frac{a - b}{|v'|}.$$

Now

$$\begin{aligned} Gf_{(III)} = {} & \frac{N}{L}\mathrm{e}^{-\beta u_1}|v|\varphi_M(v)\Big[1 - \mathrm{e}^{i(z-kv)\frac{2b}{|v|}}\Big] \\ & \times \Big\{\mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\,\Psi(v'') + \Big[1 - \mathrm{e}^{i(z-kv')\frac{a-b}{|v'|}}\Big]\Psi(v') - \Psi(v)\Big\}. \end{aligned}$$

#### 6.3.4. Region IV: $-a < x < -b$

The particle which is at the time $t = 0$ in this region and has kinetic energy less than the depth of the well, is in the bounded motion (see section 6.2). In the opposite case, the particle has undergone 1 acceleration.

If $v > \sqrt{\frac{2}{m}u_0}$ we have

$$x(-t) = x - vt\eta(t_1 - t) - [vt_1 + v'(t - t_1)]\eta(t - t_1)$$

and

$$v(-t) = v\eta(t_1 - t) + v'\eta(t - t_1)$$

with

$$v' = \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}u_0} \qquad \text{and} \qquad t_1 = \frac{x + a}{|v|}.$$

Then

$$Gf_{(IV)} = \frac{N}{L}\mathrm{e}^{\beta u_0}|v|\varphi_M(v)\left[1 - \mathrm{e}^{i(z-kv)\frac{a-b}{v}}\right][\Psi(v') - \Psi(v)].$$

#### 6.3.5. Region V: $-\infty < x < -a$

In this region the potential vanishes and operators $\exp(-tK_2)$ and $\exp(-tK_2)$ are identical, and contribution of this region to the integral operator $G$ is zero.

## 7. Kinetic equation in 1 dimension

We gather all contributions to the scattering operator found in the previous section to get KE for unbounded motions ($v^2 > \frac{2}{m}u_0$). At first we introduce common definitions of velocities appearing in the equation. These are

$$\begin{aligned}
v_1 &= \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}(u_0 + u_1)} && \text{for } v^2 \geq \frac{2}{m}(u_0 + u_1) && \text{cf. } (10)_1 \\
v_2 &= \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}u_1} && \text{for } v^2 \geq \frac{2}{m}u_1 && \text{cf. } (9)_2 \\
v_3 &= \frac{v}{|v|}\sqrt{v^2 - \frac{2}{m}u_0} && \text{for } v^2 \geq \frac{2}{m}u_0 && \text{cf. } (10)_3 \\
v_4 &= \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}u_0} && && \text{cf. } (8)_1 \\
v_5 &= \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}u_1} && && \text{cf. } (11)_2 \\
v_6 &= \frac{v}{|v|}\sqrt{v^2 + \frac{2}{m}(u_0 + u_1)} && && \text{cf. } (11)_1
\end{aligned}$$

in the form

$$(-iz + ikv)\Psi(v) - h(v) = \mathcal{G}\Psi(v)$$

with

$$h(v) = \frac{f(k, v, t = 0)}{\varphi_M(v)}$$

and

$$\begin{aligned}
&\mathcal{G}\Psi(v) \\
&= \frac{N}{L}|v| \frac{1}{\varphi_M(v)} \Big[ Gf_{(III)} + Gf_{(IV)} \\
&\quad + \eta(v^2 < \frac{2}{m}u_1) \left( Gf_{(IA)} + Gf_{(IIA)} \right) + \eta\big(v^2 > \frac{2}{m}u_1\big) \left( Gf_{(IB)} + Gf_{(IIB)} \right) \Big]
\end{aligned}$$

or

$$\begin{aligned}
&\mathcal{G}\Psi(v) \\
&= \frac{N}{L}|v| \Big[ \mathrm{e}^{-\beta u_1} \big(1 - \mathrm{e}^{i(z-kv)\frac{2b}{|v|}}\big) \Big\{ \mathrm{e}^{i(z-kv_6)\frac{a-b}{|v_6|}} \Psi(v_5) \\
&\quad + \Big[ 1 - \mathrm{e}^{i(z-kv_6)\frac{a-b}{|v_6|}} \Big] \Psi(v_6) - \Psi(v) \Big\} + \mathrm{e}^{\beta u_0} \Big[ 1 - \mathrm{e}^{i(z-kv)\frac{a-b}{v}} \Big] [\Psi(v_3) - \Psi(v)] \\
&\quad + \eta\big(\frac{2}{m}u_1 - v^2\big) \Big\{ \Big[ 1 - \mathrm{e}^{i(z-kv_4)\frac{a-b}{|v_4|}} \Big] \Psi(v_4) \\
&\quad + \Big[ 1 - \mathrm{e}^{i(z+kv_4)\frac{a-b}{|v_4|}} \Big] \mathrm{e}^{i(z-kv_4)\frac{a-b}{|v_4|}} \Psi(-v_4) + \mathrm{e}^{iz2\frac{a-b}{|v_4|}} \Psi(-v) - \Psi(v) \Big\} \\
&\quad + \mathrm{e}^{\beta u_0} \Big[ 1 - \mathrm{e}^{-i(z+kv)\frac{a-b}{|v|}} \Big] \\
&\quad \times \Big\{ \Big[ 1 - \mathrm{e}^{i(z+kv)\frac{a-b}{|v|}} \Big] \Psi(-v) + \mathrm{e}^{i(z+kv)\frac{a-b}{|v|}} \Psi(-v_3) - \Psi(v) \Big\} \\
&\quad + \eta\big(v^2 - \frac{2}{m}u_1\big) \Big\{ \Big[ 1 - \mathrm{e}^{i(z-kv_4)\frac{a-b}{|v_4|}} \Big] \Big[ 1 + \mathrm{e}^{i(z-kv_2)\frac{2b}{|v_2|}} \mathrm{e}^{i(z-kv_4)\frac{a-b}{|v_4|}} \Big] \Psi(v_4) \\
&\quad + \Big[ 1 - \mathrm{e}^{i(z+kv_2)\frac{2b}{|v_2|}} \Big] \mathrm{e}^{i(z-kv_4)\frac{a-b}{|v_4|}} \Psi(v_4) \\
&\quad - \Big[ 1 - \mathrm{e}^{i(z-kv_4)2\frac{a-b}{|v_4|}} \mathrm{e}^{i(z-kv_2)\frac{2b}{|v_2|}} \Big] \Psi(v) \Big\} \\
&\quad + \mathrm{e}^{\beta u_0} \Big[ 1 - \mathrm{e}^{i(z-kv)\frac{a-b}{|v|}} \Big] \Big\{ \Big[ 1 - \mathrm{e}^{i(z-kv_1)\frac{2b}{|v_1|}} \Big] \Psi(v_1) \\
&\quad + \mathrm{e}^{i(z-kv_1)\frac{2b}{|v_1|}} \mathrm{e}^{i(z-kv)\frac{a-b}{|v|}} \Psi(v_3) \\
&\quad + \Big( \Big[ 1 - \mathrm{e}^{i(z-kv)\frac{a-b}{|v|}} \Big] \mathrm{e}^{i(z-kv_1)\frac{2b}{|v_1|}} - 1 \Big) \Psi(v) \Big\} \Big]
\end{aligned}$$

For $k, z \to 0$, it is for the long waves and low frequencies, the scattering operator of our KE changes to the Boltzmann operator

$$G\Psi((v) = \frac{N}{L}|v| \left[ \Psi(-v) - \Psi(v) \right]. \tag{12}$$

Our scattering operator takes also the form of the Boltzmann operator for sufficiently high velocity $v$, if the time of collision of light partile with heavy particle of crystal can be neglected.

From mathematical point of view, we see that our KE generates an infinite sequence of functional equations. Its solution is a problem for the next publication.

## 8. Conclusions

We have analyzed KE valid for a dilute Lorentz gas with short range attraction potential and have given the explicit forms of the scattering operator for different forms of potential, for which some exact solutions can be found. For $k = 0$ and $z = 0$ operator reduces to the Boltzmann scattering operator. Thus our approach enlarges the possibility of description of diffusion for the case when time of particle collisions is not negligeable. The KE for light particle diffusion in one-dimensional Lorentz gas was also derived. The solution of this KE will be discussed later.

The common feature of the obtained kinetic equations is that they link the values of the probability density Fourier-Laplace transform in different points of the velocity axis. Therefore these equations are the functional equations, [91, 92].

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*Report of Meeting*

# 13th International Conference on Functional Equations and Inequalities, Małe Ciche, September 13 - 19, 2009

## Contents



The *Thirteenth International Conference on Functional Equations and Inequalities* was held from September 13 to 19, 2009 at the *Hotel Tatry* in Małe Ciche, Poland.

The series of ICFEI meetings has been organized by the *Institute of Mathematics of the Pedagogical University of Cracow* since 1984. This year the Organizing Committee consisted of Janusz Brzdęk as Chairman, Paweł Solarz, Janina Wiercioch, Władysław Wilk, and Krzysztof Ciepliński, who also acted as Scientific Secretary. The help of Jacek Chmieliński, Marek Czerni, Zbigniew Leśniak and Jolanta Olko is acknowledged with thanks.

The Scientific Committee consisted of Professors Dobiesław Brydak as Honorary Chairman, Janusz Brzdęk as Chairman, Nicole Brillouët-Belluot, Jacek Chmieliński, Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Zsolt Páles and Marek Cezary Zdun.

As usual, the conference was devoted mainly to various aspects of functional equations and inequalities. A special emphasis was given to the stability of functional equations. A *Special Session* in honor of the 100th anniversary of the birthday of Stanisław M. Ulam, devoted to this topic and chaired by Professor Themistocles M. Rassias, was held on Tuesday, September 15.

The 76 participants came from 10 countries: Austria, France, Germany, Greece, Hungary, Israel, Italy, Romania, Russia and Poland.

The conference was opened on Monday, September 14 by Professor Janusz Brzdęk — Chairman of the Scientific and Organizing Committees, who welcomed the participants in the name of the Organizing Committee and read a letter to them from Professor Władysław Błasiak, the Dean of the Faculty of Mathematics, Physics and Technical Science of the Pedagogical University. Opening address was given by Professor Jacek Chmieliński, the Director of the Institute of Mathematics. Professor Bogdan Choczewski conveyed best regards for the participants from the Honorary Chairman of the ICFEI, Professor Dobiesław Brydak. The opening ceremony was followed by the first scientific session chaired by Professor Roman Ger and the first lecture was given by Professor Gian Luigi Forti. Altogether, during 26 scientific sessions 3 lectures and 67 talks were delivered. They focused on functional equations in a single variable and in several variables, functional inequalities, stability theory, convexity, multifunctions, iteration theory, means, dynamical systems and other topics. Several contributions have been made during special *Problems and Remarks* sessions.

On Tuesday, September 15, a picnic was organized. On the next day afternoon participants visited Zakopane, the *"Winter Capital"* of Poland. The excursion included a walking tour to Strążyska Valley, Sarnia Skała and Białego Valley in the Tatra Mountains. In the evening the piano recital was performed by Marek Czerni and Hans-Heinrich Kairies. On Thursday, September 17, a banquet was held. On the following day a *Flamenco Evening* was hosted by Małgorzata Drzał (dance & vocal), Grzegorz Guzik (guitar) and Jagoda Romanowska (dance).

The conference was closed on Friday, September 18 by Professor Bogdan Choczewski. The 14th ICFEI will be organized in 2011.

The following part of the report contains abstracts of the talks (in alphabetical order of the authors' names), problems and remarks (in chronological order of presentation) and a list of participants (with addresses).

## Abstracts of Talks

**Roman Badora** *Stability of some functional equations*

Let $X$ be a group and let $\Lambda$ be a finite subgroup of the automorphism group of $X$ ($N = \operatorname{card}\Lambda$ and the action of $\lambda \in \Lambda$ on $x \in X$ is denoted by $\lambda x$). We study the stability of the following functional equations

$$\frac{1}{N}\sum_{\lambda\in\Lambda} f(x+\lambda y) = f(x)g(y) + h(y), \qquad x, y \in X,$$

$$\frac{1}{N}\sum_{\lambda\in\Lambda} f(x+\lambda y) = f(y)g(x) + h(x), \qquad x, y \in X$$

($f, g, h\colon X \to \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), which cover Jensen's functional equation, Cauchy's functional equation, the exponential functional equation, the functional equation of the square of the norm and d'Alembert's functional equation.

**Anna Bahyrycz** *On systems of equations with unknown multifunctions*

Let $(G, +)$ be a grupoid, $T$ be a nonempty set. Inspired by problem posed by Z. Moszner in [1] we investigate for which additional assumptions putting on the multifunctions $Z(t)\colon T \to 2^G$ which satisfy condition

$$\bigcup_{t\in T} Z(t) = G$$

and system of conditions

$$(\exists_{t\in T}\ i(t)j(t) \neq 0) \Longrightarrow \left(\bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)} \subset \bigcap_{t\in T} Z(t)^{i(t)j(t)}\right), \qquad (1)$$

where $Z(t)^1 := Z(t)$, $Z(t)^0 := G \setminus Z(t)$ and $i(t), j(t)\colon T \to \{0,1\}$ are the arbitrary functions not identically equal to zero, the inclusion in the above conditions (1) may be replaced by equality, obtaining the system of equations with unknown multifunctions.

[1] Z. Moszner, *Sur la fonction de choix et la fonction d'indice*, Ann. Acad. Pedagog. Crac. Stud. Math. **4** (2004), 143–169.

**Szabolcs Baják** *Invariance equations for Gini and Stolarsky means*
(joint work with **Zs. Páles**)

Given three strict means $M, N, K\colon \mathbb{R}_+^2 \to \mathbb{R}_+$, we say that the triple $(M, N, K)$ satisfies the *invariance equation* if

$$K\big(M(x,y), N(x,y)\big) = K(x,y), \qquad x, y \in \mathbb{R}_+$$

holds. It is well known that $K$ is uniquely determined by $M$ and $N$, and it is called the Gauss composition $K = M \otimes N$ of $M$ and $N$.

Our aim is to solve the invariance equation when each of the means $M, N, K$ is either a Gini or a Solarsky mean with different parameters, thus we have to consider four different equations. With the help of the computer algebra system Maple V Release 9, we give the general solutions of these equations.

**Karol Baron** *On Baire measurable solutions of some functional equations*

We establish conditions under which Baire measurable solutions $f$ of

$$\Gamma(x, y, |f(x) - f(y)|) = \Phi(x, y, f(x + \varphi_1(y)), \ldots, f(x + \varphi_N(y)))$$

defined on a metrizable topological group are continuous at zero.

**Svetlana S. Belmesova** *On the unbounded invariant curves of some polynomial maps*
(joint work with **L.S. Efremova**)

The unbounded trajectories of the quadratic mapping $F_2(x,y) = (xy, (x-2)^2)$ in the plane $\mathbb{R}^2$ has been studied in [1].

In this work we deal with the one-parameter family of the quadratic mappings

$$F_\mu(x,y) = (xy, (x-\mu)^2), \qquad (1)$$

where $(x,y) \in \mathbb{R}^2$, $\mu \in (0,1]$. It is proved the existence of the unbounded invariant curves for the mappings (1) for every $\mu \in (0,1]$.

[1] S.S. Belmesova, L.S. Efremova, *On unbounded trajectories of a certain quadratic mapping of the plane*, J. Math. Sci. (N. Y.) **157** (2009), 433–441.

**Mihály Bessenyei** *On a class of single variable functional equations*

In the last few years, functional equations have had a growing importance in competitions for secondary school students in Hungary (browse the issues of Mathematical and Physical Journal for Secondary Schools). A typical exercise is of the form

$$\alpha_1 f \circ g_1 + \ldots + \alpha_n f \circ g_n = h,$$

where $g_k, \alpha_k, h, f$ are given functions (with appropriate domain and range) under the assumption that $g_1, \ldots, g_n$ generate a group under the operation of composition. The main results of the present talk guarantee that, under some reasonable assumptions, the functional equation above (and also its nonlinear correspondence) has a unique solution. The proofs are based on Cramer's rule and the inverse-function theorem.


[1] *Mathematical and Physical Journal for Secondary Schools (KöMal)* (http://www.komal.hu).

[2] V.S. Brodskii, A.K. Slipenko, *Functional equations*, Visa Skola, Kiev, 1986 (in Russian).

[3] K. Lajkó, *Functional equations in exercises*, University Press of Debrecen, 2005 (in Hungarian).


**Zoltán Boros** *Inequalities for pairs of additive functions*

Representation theorems are presented for pairs of additive functions, under the assumption that a related expression is locally bounded. Let us assume that $f$ and $g$ are real additive functions. If

$$\frac{1}{x} f(x) + x g\Big(\frac{1}{x}\Big)$$

is bounded on a non-void open interval or

$$x f(x) + \sqrt{1-x^2}\, g\big(\sqrt{1-x^2}\big)$$

is bounded on every compact subinterval of the open interval $(0, 1)$, then there exists a real derivation $d$ such that

$$f(x) = d(x) + f(1)x \quad \text{and} \quad g(x) = d(x) + g(1)x$$

for every real number $x$. However, if, for instance,

$$\sqrt{1-x^2}\, f(x) - x g\big(\sqrt{1-x^2}\big)$$

is bounded on every compact subinterval of the open interval $(0, 1)$, then $f$ and $g$ are linear.

**Nicole Brillouët-Belluot** *Some further results concerning a conditional Gołąb–Schinzel equation*
(joint work with **J. Chudziak** and **J. Brzdęk**)

Let $X$ be a real linear space and let $M : \mathbb{R} \to \mathbb{R}$ be a continuous and multiplicative function. We determine the solutions $f : X \to \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y) f(x) f(y)[f(x + M(f(x))y) - f(x)f(y)] = 0$$

which are continuous on rays, i.e., which are such that, for every $x \in X \setminus \{0\}$, $f_x: \mathbb{R} \to \mathbb{R}$ defined by $f_x(t) = f(tx)$ is continuous.

In the particular cases where $M \equiv 1$ and $M(x) \equiv x$, we obtain the continuous on rays solutions of a conditional exponential equation and those of a conditional Gołąb–Schinzel equation.

These results extend those given by the authors at the 47th ISFE in Gargnano.

**Janusz Brzdęk** *On nonstability of the linear recurrence of order one*
(joint work with **D. Popa** and **B. Xu**)

Let $\mathbb{K}$ be either the field of reals or the field of complex numbers, $X$ be a Banach space over $\mathbb{K}$, $(a_n)_{n\geq 0}$ a sequence in $\mathbb{K} \setminus \{0\}$, and $(b_n)_{n\geq 0}$ a sequence in $X$. We present a result concerning nonstability of the linear recurrence

$$y_{n+1} = a_n y_n + b_n, \qquad n \geq 0.$$

This corresponds to the contents, e.g., of recent papers [1–5].


[1] J. Brzdęk, D. Popa, B. Xu, *Note on nonstability of the linear recurrence*, Abh. Math. Sem. Univ. Hamburg **76** (2006), 183–189.

[2] J. Brzdęk, D. Popa, B. Xu, *The Hyers–Ulam stability of nonlinear recurrences*, J. Math. Anal. Appl. **335** (2007), 443–449.

[3] J. Brzdęk, D. Popa, B. Xu, *Hyers–Ulam stability for linear equations of higher orders*, Acta Math. Hungar. **120** (2008), 1–8.

[4] D. Popa, *Hyers–Ulam–Rassias stability of a linear recurrence*, J. Math. Anal. Appl. **309** (2005), 591–597.

[5] T. Trif, *On the stability of a general gamma-type functional equation*, Publ. Math. Debrecen **60** (2002), 47–61.


**Pál Burai** *Some results on Orlicz-convex functions*
(joint work with **A. Házy**)

Let $X$ be a linear space over the real field $\mathbb{R}$, and $\mathcal{C} \subset X$ be an open, nonempty cone. A function $f: \mathcal{C} \to \mathbb{R}$ is called $s$-convex (Orlicz-convex) if

$$f\left(\lambda^s x + (1-\lambda)^s y\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \mathcal{C}$, $\lambda \in (0,1]$, where $s \in [1,\infty)$ is a fixed number. In this talk we make some examination in this class of functions.

**Liviu Cădariu** *Remarks on the fixed point method for Ulam–Hyers stability*

In [1] and [2] some generalized Ulam–Hyers stability results for Cauchy functional equation have been proved. One of the results reads as follows:

*Let us consider a real linear space $E$, a complete $p$-normed space $F$ and a subhomogenous functional of order $\alpha$ $\|(\cdot,\cdot)\|_\alpha: E \times E \to [0,\infty)$, with $\alpha \neq p$. In these conditions, the following stability property holds: For each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every mapping $f: E \to F$ which satisfies*

$$\|f(x) + f(y) - f(x+y)\|_p \leq \delta(\varepsilon) \cdot \|(x,y)\|_\alpha, \qquad x, y \in E,$$

*there exists a unique additive mapping* $a: E \to F$ *such that*

$$||f(x) - a(x)||_p \le \varepsilon \cdot ||(x,x)||_\alpha, \qquad x \in E.$$

We intend to outline the results concerning the generalized Ulam–Hyers stability for different other kinds of functional equations. Both the Hyers direct method and the fixed point method will be emphasized and we shall consider functions defined on linear spaces and taking values in $p$-normed spaces or random normed spaces.

[1] L. Cădariu, *A general theorem of stability for the Cauchy's equation*, Bull. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz. **47(61)** (2002), 14–28.

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed points approach*, Iteration theory (ECIT'02), 43–52, Grazer Math. Ber. **346**, Karl–Franzens–Univ. Graz, Graz, 2004.

[3] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[4] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

**Jacek Chmieliński** *Stability of linear isometries and orthogonality preserving mappings*

In reference to a question posed by the author during the 12th ICFEI, a short survey on *linear approximate isometries* in normed spaces and respective stability problems will be given.

Next, an application to the problem of stability of orthogonality preserving mappings in normed spaces will be shown. Results from a joint work with **P. Wójcik** will be presented.

**Jacek Chudziak** *Stability of a composite functional equation*

At the 47th International Symposium on Functional Equations (Gargnano, Italy) J. Brzdęk has posed several questions concerning a quotient stability of the following generalization of the Gołąb–Schinzel functional equation

$$f(x + M(f(x))y) = f(x)f(y).$$

In our talk we present the answers for some of them.

**Krzysztof Ciepliński** *Stability of the multi-Jensen equation*

Assume that $V$ is a normed space, $W$ is a Banach space and $m \ge 2$ is an integer. A function $f: V^m \to W$ is called *multi-Jensen* (we also say that $f$ satisfies multi-Jensen equation) if it is a Jensen mapping in each variable, that is

$$\begin{aligned} &f(x_1,\dots,x_{i-1},\tfrac{1}{2}(x_i+y_i),x_{i+1},\dots,x_m)\\ &\quad= \frac{1}{2}f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m) + \frac{1}{2}f(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m),\\ &\qquad\qquad i \in \{1,\dots,m\},\ x_1,\dots,x_i,y_i,\dots,x_m \in V. \end{aligned}$$

This notion was introduced by W. Prager and J. Schwaiger in 2005 with the connection with generalized polynomials (see [1]).

In this talk the stability of multi-Jensen equation is discussed.

[1] W. Prager, J. Schwaiger, *Multi-affine and multi-Jensen functions and their connection with generalized polynomials*, Aequationes Math. **69** (2005), 41–57.

**Stefan Czerwik** *S.M. Ulam – his life and results in mathematics, physics and biology*

We shall present the information about the life of S.M. Ulam and his results in different areas of science: mathematics, physics and biology; particularly in stability of functional equations and H-bomb.

**Zoltán Daróczy** *On an elementary inequality and conjugate means*
(joint work with **Zs. Páles**)

Let $n \geq 2$, $k \geq 1$. In this talk we give the necessary and sufficient condition for the real numbers $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_k$ to fulfill the following property:

If

$$\min\{x_i\} \leq M_l \leq \max\{x_i\}, \qquad l = 1, 2, \dots, k$$

holds for all real numbers $x_1, x_2, \dots, x_n$ and $M_1, M_2, \dots, M_k$, then

$$\min\{x_i\} \leq \sum_{i=1}^{n} p_i x_i + \sum_{l=1}^{k} q_l M_l \leq \max\{x_i\}.$$

Let $I$ be a nonvoid open interval and let $M_l\colon I^n \to I$ $(l = 1, 2, \dots, k)$ be means. If there exist $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_k$ with the property above and a strictly monotone, continuous function $\varphi$ on $I$, then

$$M(x_1, x_2, \dots, x_n) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i) + \sum_{l=1}^{k} q_l \varphi(M_l(x_1, x_2, \dots, x_n))\right), \qquad x_1, x_2, \dots, x_n \in I$$

is a mean value and we call it the *conjugate mean generated by the means* $M_1$, $M_2, \dots, M_k$.

We deal with several problems on conjugate means.

**Judita Dascăl** *On conjugate means*
(joint work with **Z. Daróczy**)

Let $I \subset \mathbb{R}$ be a nonvoid open interval.

A function $M\colon I^2 \to I$ is said to be a *conjugate mean* on $I$ if there exist real numbers $p, q \in [0, 1]$ and a continuous, strictly monotone real valued function $\varphi$ defined on $I$ such that

$$M(x, y) = \varphi^{-1}\Big(p\varphi(x) + q\varphi(y) + (1 - p - q)\varphi\Big(\frac{x + y}{2}\Big)\Big), \qquad x, y \in I.$$

We deal with the equality problem in the class of conjugate means.

**Joachim Domsta** *A comparison of quantum dynamical semigroups obtainable by mixing or partial tracing*

Some simple examples of quantum systems are collected to illustrate requirements sufficient for the evolution of a subsystem according to a quantum dynamical semigroup. For this, a class of quantum dynamics of a system $S$ coupled to a reservoir $R$ is analyzed in the Hilbert space $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$, where $\mathcal{H}_R = L^2(\mathbb{R})$ and $\mathcal{H}_S = l_I^2$, with $I$ standing for a complete at most countable set of pure orthogonal states of $S$. The Hamiltonian of $SR$ is built of tensor products of multipliers acting on $\mathcal{H}_S$ and $\mathcal{H}_R$. The chosen linear coupling implies the exponential decoherence of the reduced evolution of $S$ if and only if the occupation density in $R$ is of the Cauchy type. Then the system indicates the exponential decoherence. On the other hand, since the occupation density in $S$ is discrete, the reduced evolution of $R$ is never governed by a semigroup (unless there is no coupling).

In the considered case, the reduced evolution of the subsystem $S$ as well as of the reservoir $R$ can be equivalently obtained by taking the expectation (i.e. by averaging) of the unitary dynamics of the alone standing system $S$ or $R$ with suitably chosen random Hamiltonians. Thus again, the probability distribution of the random perturbation for $S$ must be of the Cauchy type if the exponential decoherence should follow.

In the models of the third class the phase of the quantum system $S$ varies according to a stochastic process with independent stationary increments. In other words, this is an example of a random dynamical system. Then the exponential decoherence of the evolution of the averaged state follows, independently of the distribution of the process. In such cases the Itô-Schrödinger equation for the random unitary dynamics and the master equation for the averaged density matrices are obtained in the dependence on the probability distribution of the process. For presenting the Cauchy distribution in a different context, a relation to the exponential decay of the autocorrelation of autonomous systems is discussed briefly.

**Andrey S. Filchenkov** *On the simplest topologically transitive skew products in the plane*
(joint work with **L.S. Efremova**)

Let $F(x,y) = (f(x), g_x(y)) : I \to I$ be a skew product of interval maps, $I$ is a rectangle in the plane, $I = I_1 \times I_2$ ($I_1$, $I_2$ are closed intervals). Let $T^1(I)$ be the space of $C^1(I)$-smooth skew products of interval maps.

In this talk we present conditions of the density of the set of periodic points in the phase space of the skew product.

Theorem.
*Let $F \in T^1(I)$ satisfy the following conditions:*

1) *$F(x,y)$ is a topologically transitive skew product of interval maps,*

2) *the partial derivative $\frac{\partial g_x(y)}{\partial y}$ monotonically decreases with respect to $y \in I_2$ for any $x \in I_1$,*

3) *$g_x(\partial I_2) = \partial I_2$ for any $x \in I_1$, where $\partial I_2$ is the boundary of $I_2$.*

*Then the set of periodic points of the skew product of interval maps is dense in $I$.*

In this talk we also construct the topologically transitive skew product which satisfies all conditions of the above theorem. We use here the unimodal maps theory (see [2]). For the comparison in [3] it is proved the existence of the topologically transitive cylindrical cascade (the skew product over the irrational rotation of the circle) without periodic points. In [1] it is constructed an example of continuous but not smooth topologically transitive skew product in the unit square which has the dense set of periodic points in horizontal fibers $y = 0$ and $y = 1$.

[1] Ll. Alseda, S. Kolyada, J. Llibre, L. Snoha, *Entropy and periodic points for transitive maps*, Trans. Amer. Math. Soc. **351** (1999), 1551–1573.

[2] L.S. Efremova, A.S. Filchenkov, *About one example of the topologically transitive skew product of interval maps in the plane*, Math problems, M.:MPhTI 2009, 61–68.

[3] E.A. Sidorov, *Topologically transitive cylindrical cascades* (Russian), Mat. Zametki **14** (1973), 441–452.

**Gian Luigi Forti** *Symbolic dynamics generated by graphs*

In many natural phenomena strings consisting of sequences of symbols play a central role. Also the evolution of large classes of dynamical systems can be described, under certain conditions, as a sequence of symbols. In this context, a central question is how to enumerate and to characterize the full set of possible sequences generated by a dynamical system.

At first, the properties of the symbolic dynamics generated by a graph on an alphabet are presented and it is shown that the number of sequences of length $n$ is either exponential or polynomial with respect to $n$.

Then by a combination of several graphs we obtain different laws. In particular we can obtain laws observed in complex systems and conjectured in 1992 by Ebeling and Nicolis.

We finish by presenting a probabilistic approach to the problem.

[1] V. Basios, G.-L. Forti, G. Nicolis, *Symbolic dynamics generated by a combination of graphs*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **18** (2008), 2265–2274.

**Roman Ger** *On a problem of Cuculière*

In the February 2008 issue of *The American Mathematical Monthly* (Problems and Solutions, p.166) the following question was proposed by R. Cuculière:

*Find all nondecreasing functions $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that*
$$f(x + f(y)) = f(f(x)) + f(y) \text{ for all real } x \text{ and } y$$

(Problem **11345**).
We shall present:

- the general Lebesgue measurable solution,
- monotonic solutions,
- a description of the general solution

of the functional equation in question.

**Attila Gilányi** *Conditional stability of monomial functional equations*

During the 42nd International Symposium on Functional Equations in Opava, Czech Republic, 2004, J. Aczél announced the program of the investigation of conditional functional equations (c.f. [1]). Connected to this program, we present some conditional stability results for monomial functional equations.

More precisely, in the case of various sets $D \subseteq \mathbb{R}\times\mathbb{R}$ and $H \subseteq \mathbb{R}$, and assuming that $Y$ is a Banach space, $n$ is a positive integer, $\alpha$ is an arbitrary, $\varepsilon$ and $\delta$ are nonnegative real numbers, we examine whether the validity of the inequality

$$\left\|\Delta_y^n f(x) - n!f(y)\right\| \leq \varepsilon|x|^\alpha + \delta|y|^\alpha, \qquad (x,y) \in D$$

implies the existence of nonnegative constants $c$ and $d$ and a monomial function $g\colon \mathbb{R} \to Y$ of degree $n$ (i.e. a solution of the functional equation $\Delta_y^n g(x) - n!g(y) = 0$, $x, y \in \mathbb{R}$) for which

$$\|f(x) - g(x)\| \leq (c\varepsilon + d\delta)|x|^\alpha, \qquad x \in H$$

holds.

[1] J. Aczél, *5. Remark, Report of Meeting*, Aequationes Math. **69** (2005), 183.

**Dorota Głazowska** *An invariance of the geometric mean with respect to the Cauchy mean-type mappings*
(joint work with **J. Matkowski**)

We consider the problem of invariance of the geometric mean with respect to the Cauchy mean-type mappings $\left(D^{f,g}, D^{h,k}\right)$, i.e., the functional equation

$$G \circ \left(D^{f,g}, D^{h,k}\right) = G.$$

Assuming that the generators $g$ and $k$ are power functions we show that the functions $f$ and $h$ have to be of high class of regularity. This fact allows to reduce the problem to differential equations and find some necessary conditions for generators $f$ and $h$.

**Eszter Gselmann** *On the stability of derivations*

In this talk we investigate the stability of a system of functional equations that defines real derivations. More precisely, the problem of Ulam is considered in connection with the following system of equations

$$f(x+y) = f(x) + f(y), \qquad x \in \mathbb{R}$$

and

$$f(x^n) = cx^k f(x^m), \qquad x \in \mathbb{R} \setminus \{0\},$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is the unknown function, $c \in \mathbb{R}$ and $n, m, k \in \mathbb{R}$ are arbitrarily fixed. Using a preliminary lemma that is also presented, it is proved that the above system of functional equations is stable in the sense of Hyers and Ulam, under some conditions on the parameters $c, n, m$ and $k$.

**Grzegorz Guzik** *On some disjoint iteration semigroups on the torus*

General construction of measurable (continuous) disjoint iteration semigroups of triangular mappings on the torus is given.

**Attila Házy** *Bernstein–Doetsch type results for h-convex functions*

The concept of $h$-convexity was introduced by S. Varošanec in [1]. In our talk we introduce a more general concept of the $h$-convexity, and the concept of the so called $(H, h)$-convexity.

A $h$-convex (or $(H, h)$-convex) function is defined as a function $f\colon D \to \mathbb{R}$ (where $D$ is a nonempty, open, convex subset of a real (or complex) linear space) which satisfies

$$f(\lambda x + (1-\lambda)y) \le h(\lambda)f(x) + h(1-\lambda)f(y),$$

for all $x, y \in D$ and $\lambda \in [0,1]$ (resp. $\lambda \in H$), where $h$ is a given real function.

The main goal of our talk is to prove some regularity and Bernstein–Doetsch type result for $h$-convex and $(H, h)$-convex functions. We also collect some facts on such functions. Finally, we collect some interesting, easily-proved properties of $h$-convex functions.

[1] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

**Eliza Jabłońska** *About solutions of a generalized Gołąb–Schinzel equation*

Let $n \in \mathbb{N}$ and let $X$ be a metrizable linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We consider solutions $f\colon X \to \mathbb{K}$ of the functional equation

$$f(x + f(x)^n y) = f(x)f(y) \qquad \text{for } x, y \in X$$

such that either $f$ is bounded on a set of second category with the Baire property or $f$ is Baire measurable. Our result generalizes a result of J. Brzdęk.

**Hans-Heinrich Kairies** *A sum type operator*

Our sum type operator $F\colon D \to F[D]$ is given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x),$$

where $D = \{\varphi\colon \mathbb{R} \to \mathbb{R} : \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x) \text{ converges for every } x \in \mathbb{R}\}$.

We treat the following aspects:

1. Historical background.
2. Basic properties of $F$ and its restrictions $F_{rg}\colon D_{rg} \to F[D_{rg}]$ to sixteen subspaces $D_{rg}$ of $D$, which are all vector spaces and in part Banach spaces.
3. Functional equations for $F[\varphi]$ and characterizations.
4. Some Fourier analysis for $F[\varphi]$.

5. Images $F[S]$ and $F^{-1}[S]$.
6. Eigenvalues and eigenspaces for all the sixteen $F_{rg}$.
7. Continuous and residual spectra.
8. Extensions.

**Barbara Koclęga-Kulpa** *On a class of equations stemming from various quadrature rules*
(joint work with **T. Szostok**)

We deal with a functional equation of the form

$$F(y) - F(x) = (y - x)\sum_{k=1}^{n} a_k f(\lambda_k x + (1 - \lambda_k)y), \qquad x, y \in \mathbb{R} \tag{1}$$

motivated by quadrature rules of approximate integration. In previous results the solutions of this equation were found only in some particular cases. For example, coefficients $\lambda_k$ were supposed to be rational or the equation in question was solved only for $n = 2$.

We prove that every function $f: \mathbb{R} \to \mathbb{R}$ satysfying equation (1) with some function $F: \mathbb{R} \to \mathbb{R}$, where $\sum_{k=1}^{n} a_k \neq 0$, is a polynomial of degree at most $2n - 1$. In our results we do not assume any specific form of coefficients occuring at the right-hand side of (1) and we allow $n$ to be any positive integer. Moreover, we obtain solutions of our equation without any regularity assumptions concerning functions $f$ and $F$.

**Zygfryd Kominek** *On a Jensen–Hosszú equation*
(joint work with **J. Sikorska**)

It is known that in the class of functions acting the interval $I = [0, 1]$ ($I = (0, 1)$) into a real Banach space the Jensen functional equation is stable and the Hosszú functional equation has not this property. So, we have a nontrivial pair of the equivalent equations such that one of them is stable and the other is not. From this point of view it seems interesting to consider the functional equation of the form

$$f(x + y - xy) + f(xy) = 2f\Big(\frac{x + y}{2}\Big), \qquad x, y \in I. \tag{1}$$

The left-hand-side of equation (1) is the same as the left-hand-side of the Hosszú functional equation, and the right-hand-side of our equation coincides with the left-hand side of the Jensen equation. We will prove that equation (1) is also equivalent to the Jensen (and in the same reason to the Hosszú) equation and, moreover, that equation (1) is stable in the sense of Hyers and Ulam.

**Dorota Krassowska** *On iteration semigroups containing generalized convex and concave functions*

Let $I \subset \mathbb{R}$ be an open interval and let $M, N: I^2 \to I$ be continuous functions. A function $f: I \to I$ is said to be $(M, N)$-*convex* ($(M, N)$-*concave*) if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)), \qquad x, y \in I.$$

A function $f\colon I \to I$ simulteneously $(M,N)$-convex and $(M,N)$-concave is called $(M,N)$-*affine* (see [1]).

We prove that if in a continuous iteration semigroup $\{f^t,\ t \geq 0\}$ every element $f^t$ is $(M,N)$-convex or $(M,N)$-concave and there exist $r > s > 0$ such that $f^r$ and $f^s$ are $(M,N)$-affine, then $M = N$ and every element of a semigroup is $(M,M)$-affine. We also consider the case where $M = N$ and we show that if in a continuous iteration semigroup $\{f^t,\ t \geq 0\}$ there exist $f^r < \mathrm{id}$ and $f^s < \mathrm{id}$ such that $\frac{r}{s} \notin \mathbb{Q}$ and $f^r$ is $(M,M)$-convex and $f^s$ is $(M,M)$-concave, then every element of the semigroup is $(M,M)$-affine.

[1] J. Matkowski, *Iteration groups with generalized convex and concave elements*, Iteration theory (ECIT 94) (Opava), 199–216, Grazer Math. Ber. **334**, Karl-Franzens-Univ. Graz, Graz, 1997.

**Zbigniew Leśniak** *On conjugacy of Brouwer homeomorphisms*

We consider Brouwer homeomorphisms of the plane for which the oscillating set is empty. The main result says that if the sets of indices of coverings of the plane consisting of maximal parallelizable regions for two Brouwer homeomorphisms are isomorphic and if for each of these regions there exists a one-to-one correspondence between the set of singular lines contained in the boundary of the region and the set of singular lines contained in the interior of the region, then these Brouwer homeomorphisms are conjugated. This theorem holds for Brouwer homeomorphisms that are embeddable in a flow as well as for Brouwer homeomorphisms for which there exists a foliation of the plane consisting of invariant topological lines.

**Andrzej Mach** *Stability of some functional equations and open problems*
(joint work with **Z. Moszner**)

Some results on stability of certain equations and systems of equations are given. A number of open problems of stability, raised by Z. Moszner, is presented. The answer for one of them is given.

[1] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A **27** (1941), 222–224.

[2] A. Mach, *On some functional equations involving Babbage equation*, Results Math. **51** (2007), 97–106.

[3] A. Mach, Z. Moszner, *On stability of the translation equation in some classes of functions*. Aequationes Math. **72** (2006), 191–197.

[4] A. Mach, Z. Moszner, *On some functional equations involving involutions*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **216** (2007), 3–13 (2008).

[5] Z. Moszner, *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88.

**Ewelina Mainka** *On uniformly continuous Nemytskii operators generated by set-valued functions*

Let $I = [0,1]$, let $Y$ be a real normed linear space, $C$ a convex cone in $Y$ and $Z$ a Banach space. Denote by $\mathrm{clb}(Z)$ the set of all nonempty closed and bounded subsets of $Z$.

If a superposition operator $N$ generated by a set-valued function $F\colon I\times C\to \mathrm{clb}(Z)$ maps the set $H_\alpha(I,C)$ of all functions $\varphi\colon I\to C$ satisfying the Hölder condition into the set $H_\beta(I,\mathrm{clb}(Z))$ of all set-valued functions $\phi\colon I\to\mathrm{clb}(Z)$ satisfying the Hölder condition and is uniformly continuous, then

$$F(x,y)=A(x,y)\overset{*}{+}B(x),\qquad x\in I,\ y\in C$$

for some set-valued functions $A,B$ such that $A(\cdot,y),B\in H_\beta(I,\mathrm{clb}(Z))$, $y\in C$ and $A(x,\cdot)\in\mathcal{L}(C,\mathrm{clb}(Z))$, $x\in I$.

Using Jensen functional equation is essential in the proof. A converse result is also considered.

**Judit Makó** *On $\varphi$-convexity*
(joint work with **Zs. Páles**)

In this talk a new concept of approximate convexity is definied, termed *$\varphi$-convexity.* The function $\varphi$ is chosen in a particular way. Assume that $I$ is a nonempty open real interval of $\mathbb{R}$ and denote $I^*:=(I-I)\cap\mathbb{R}_+$, where $\mathbb{R}_+$ stands for the set of nonnegative real numbers. Let $\varphi\colon I^*\to\mathbb{R}_+$ be a given function. A real valued function $f\colon I\to\mathbb{R}$ is called *$\varphi$-convex* if

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)+t\varphi\big((1-t)|x-y|\big)+(1-t)\varphi\big(t|x-y|\big)\quad(1)$$

for all $t\in[0,1]$ and for all $x,y\in I$. If (1) holds for $t=\frac12$, then we say that $f$ is *$\varphi$-midconvex.*

In this talk we give some equivalent conditions for $\varphi$-convexity. Furthermore, we search relations between the local upper-bounded $\varphi$-midconvex functions and $\varphi$-convex functions.

**Gyula Maksa** *Nonnegative information functions revisited*
(joint work with **E. Gselmann**)

Motivated by the known result that there are nonnegative information functions different from the Shannon information function, in this talk, we present some properties of the set on which every nonnegative information function coincides with the Shannon's one.

**Fruzsina Mészáros** *Density function solutions of a functional equation*
(joint work with **K. Lajkó**)

The functional equation

$$f_U(u)\,f_V(v)=f_X\Big(\frac{1-v}{1-uv}\Big)f_Y(1-uv)\,\frac{v}{1-uv}$$

is investigated for almost all $(u,v)\in(0,1)^2$. Suppose only that the unknown functions $f_X,f_Y,f_U,f_V\colon(0,1)\to\mathbb{R}$ are density functions of some random variables (i.e. nonnegative and Lebegue integrable with integral 1). Does it follow that they are positive almost everywhere on $(0,1)$?

Using a method of A. Járai in connection with the characterization of the Dirichlet distribution, we give an affirmative answer to this question.

The obtained result is related to an independence property for beta distributions.

**Bartosz Micherda** *On the properties of four elements in function spaces*

Let $X_\rho$ be a modular space which is a lattice with respect to the ordering $\geq$ given by some pointed convex cone $K \subset X_\rho$. For $x, y \in X_\rho$ denote $x \wedge y = \inf(x, y)$ and $x \vee y = \sup(x, y)$.

Then we say that $\rho$ satisfies *the lower property of four elements* ($LPFE$) if for any $x, y, w, z \in X_\rho$ such that $x \geq y$, we have

$$\rho(x - w) + \rho(y - z) \geq \rho(x - w \vee z) + \rho(y - w \wedge z),$$

and it satisfies *the upper property of four elements* ($UPFE$) if for any $x, y, w, z \in X_\rho$ such that $x \geq y$, we have

$$\rho(x - w) + \rho(y - z) \leq \rho(x - w \wedge z) + \rho(y - w \vee z).$$

These inequalities are useful for the study of projection and antiprojection operators in modular spaces (see [1] and [2]).

In our talk we present a class of function modulars which satisfy both (LPFE) and (UPFE). We also give some other examples and counterexamples.

[1] G. Isac, G. Lewicki, *On the property of four elements in modular spaces*, Acta Math. Hungar. **83** (1999), 293–301.

[2] B. Micherda, *The properties of four elements in Orlicz-Musielak spaces*, Math. Inequal. Appl. **4** (2001), 599–608.

**Vladimir Mityushev** *Application of functional equations to determination of the effective conductivity of composites with elliptical inclusions*

Analysis concerning the transport properties of inhomogeneous materials is of fundamental theoretical interest. Analytical formulae for the macroscopic properties with physical and geometrical parameters in symbolic form is useful to predict the behavior of composites. The method of functional equations is one of the constructive methods to derive such analytical exact and approximate formulae. The present talk is devoted to application of the method to two–dimensional composites with elliptical inclusions. The sizes, the locations and the orientations of the ellipses can be arbitrary. The analytical formulae contains all above geometrical parameters in symbolic form.

**Lajos Molnár** *Characterizing some specific elements in spaces of operators and functions and its use*

We characterize certain specific elements in spaces of functions or Hilbert space operators and use those characterizations to determine the structures of different kinds of automorphisms and isometries of the underlying spaces.

**Janusz Morawiec** *Refinement equations and Markov operators*
(joint work with **R. Kapica**)

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, let $L\colon \Omega \to \mathbb{R}^n$ be a random vector and let $K\colon \Omega \to \mathbb{R}^{n\times n}$ be a random matrix. We discuss the close connection between the problem of the existence of non-trivial $L^1$-solutions $f\colon \mathbb{R}^n \to \mathbb{R}$ of the refinement equation

$$f(x) = \int_{\Omega} |\det K(\omega)| f(K(\omega)x - L(\omega))\, dP(\omega)$$

and the problem of the existence of invariant probability Borel measures of a very special Markov operator defined (on the space of all finite Borel measures on $\mathbb{R}^n$) by

$$M\mu(A) = \int_{\Omega}\int_{\mathbb{R}^p} \chi_A(K(\omega)^{-1}(x + L(\omega)))d\mu(x)\, dP(\omega).$$

**Jacek Mrowiec** *On stability of some functional equation*

Recently, Soon–Mo Jung has proved the Hyers–Ulam stability of the Fibonacci functional equation

$$f(x) = f(x-1) + f(x-2)$$

in the class of functions $f: \mathbb{R} \to X$, where $X$ is a real Banach space. The same method with little modifications may be applied to prove stability of the more general functional equation

$$f(x) = af(x-1) + bf(x-2),$$

where $a, b \in \mathbb{R}$, in the same class of functions. However, for some values of $a$ and $b$ this equation is not stable.

**Anna Mureńko** *A generalization of Bernstein–Doetsch theorem*

Let $V$ be an open convex subset of a nontrivial real normed space $X$. We give a partial generalization of Bernstein–Doetsch theorem. Namely, if there exist a base $\mathcal{B}$ of $X$ and a point $x \in V$ such that a midconvex function $f: X \to \mathbb{R}$ is locally bounded above on $b$-ray at $x$ for each $b \in \mathcal{B}$, then $f$ is convex. Moreover, under the above assumption, $f$ is also continuous in case $X = \mathbb{R}^N$, but not in general.

**Adam Najdecki** *On stability of some functional equation*

Let $S$ be a nonempty set, $k, n \in \mathbb{N}$ and $g_j: S \times S \to S$ for $j \in \{1, \ldots, k\}$. We are going to discuss the stability of the functional equation

$$\sum_{j=1}^{k} f(g_j(s,t)) = f(s)f(t), \qquad s, t \in S$$

in the class of functions $f$ from $S$ to the normed algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices.

**Kazimierz Nikodem** *Remarks on strongly convex functions*

Let $D$ be a convex subset of a normed space and $c > 0$. A function $f: D \to \mathbb{R}$ is called *strongly convex* with modulus $c$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0,1]$. We say that $f$ is *midpoint strongly convex* with modulus $c$ if

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \qquad x,y \in D.$$

Some properties of midpoint strongly convex functions (corresponding to the classical results of Jensen convex functions) are presented. A relationship between strong convexity and generalized convexity in the sense of Beckenbach is also given.

**Andrey A. Nuyatov** *Representation of space of entire functions of Fischer's pairs*

In [2] resolvability of the equation

$$\psi_1(z)M_{F_1}[f] + \ldots + \psi_m(z)M_{F_m}[f] = g(z) \tag{1}$$

is proved, $\overrightarrow{\psi} = (\psi_1(z), \ldots, \psi_m(z)) \in H^m_{C^n}$, $M_{F_j}[f] \equiv (F_j, f(z+w))$ – the operator of convolution in the space $H(C^n)$, which characteristic function is equal to $\varphi_j(z)$, $j = 1, \ldots, m$. Resolvability of this equation is connected by concept of Fisher's pairs (see [1]):

A pair of polynomials $(P(z), Q(D))$, $D = (D_1, \ldots, D_n)$, $D_j = \partial/\partial z_j$ forms a *Fischer pair* if

$$H(C^n) = (P(z)) \oplus \mathrm{Ker}Q(D).$$

In this connection, equation (1) can be written down in the following way

$$\Sigma^0_{k=m} P_k(z) M_{P^*_k}[f] = g(z), \tag{2}$$

where $\deg P_k = \deg P^*_k = k$, $k = 0, \ldots, m$. Equation (2) will become

$$\Sigma^0_{k=m} \big(\Sigma^0_{|\alpha|=k} a^k_\alpha z^\alpha\big)\big(\Sigma^0_{|\alpha|=k} \overline{a}^k_\alpha D^\alpha f\big) = g(z). \tag{3}$$

We will show under what conditions the differential equation with variable factors

$$\Sigma^0_{|\beta|=m}\big[\big(\Sigma^0_{|\alpha|=m} b_{\alpha\beta} z^\alpha\big) D^\beta f\big] = g(z) \tag{4}$$

is led to equation (3), i.e., the factors of equation (3) are expressed through the factors of equation (4). Let $B = ||b_{\alpha\beta}||$ be matrix of factors of equation (4).

Theorem.
*If the transposed matrix to B can be represented in the form of* $B^T = \Sigma^0_{k=m} B_k$, *where* $B_k = ||b^k_{\alpha\beta}||$ $(k = m, m-1, \ldots, 0)$ *- Hermitean conjugate matrixes of a rank 1, thus the only elements of the last of* $\frac{1}{(n-1)!}\sum_{i=0}^k \prod_{j=1}^{n-1}(i+j)$, $n \geq 2$ *rows and* $\frac{1}{(n-1)!}\sum_{i=0}^k \prod_{j=1}^{n-1}(i+j)$, $n \geq 2$ *columns are nonzero, then equation* (4) *is led to equation* (3).

The program which checks conditions of reduction of the given equation to equation (3) and if it is possible is written and expresses the factors of equation (3) through the factors of equation (4) and writes down equation (3).

[1] H.S. Shapiro, *An algebraic theorem of E. Fischer, and the holomorphic Goursat problem*, Bull. London Math. Soc. **21** (1989), 513–537.

[2] V.V. Napalkov, *On the theory of linear differential equations with variable coefficients* (Russian), Dokl. Akad. Nauk **397** (2004), 748–750.

**Andrzej Olbryś** *On some inequality connected with Wright convexity*

We consider the functional inequality

$$f(\lambda x+(1-\lambda)y) \le G(x,y,\lambda)f(x)+[1-G(x,y,\lambda)]f(y), \quad x,y\in(a,b),\ \lambda\in(0,1),$$

where $f\colon(a,b)\to\mathbb{R}$ and $G\colon(a,b)\times(a,b)\times(0,1)\to\mathbb{R}$ is a function symetric with respect to $x$ and $y$.

**Jolanta Olko** *On a family of multifunctions*

Let $\{f^t,\ t\in\mathbb{R}\}$, $\{g^t,\ t\in\mathbb{R}\}$ be groups of increasing selfmappings of an interval $I$ such that $f^t\le g^t$, $t\in\mathbb{R}$. We study properties of the family $\{H^t,\ t\in\mathbb{R}\}$ of multifunctions defined as follows

$$H^t(x)=[f^t(x),g^t(x)], \qquad x\in I,\ t\in\mathbb{R}.$$

**Zsolt Páles** *An application of Blumberg's theorem in the comparison of weighted quasi-arithmetic means*

We present comparison theorems for the weighted quasi-arithmetic means and for weighted Bajraktarević means without supposing in advance that the weights are the same. The results have been obtained jointly with Gyula Maksa under differentiability assumptions. Using Blumberg's theorem (stating, for every real function, the existence of a countable dense set such that the restriction of the function to this set is continuous), these regularity assumptions are completely removed.

**Boris Paneah** *Several remarks on approximate solvability of the linear functional equations*

We consider the general linear functional operator

$$\mathcal{P}F(x) := \sum_{j=1}^{N} c_j(x)F\circ a_j(x), \qquad x\in D\subset\mathbb{R}^p.$$

Here $F\in C(I,B)$ (the space of all $B$-valued continuous functions on $I$) with $I=(-1,1)$, $B$ a Banach space, *coefficients* $c_j$ and *arguments* $a_j$ of $\mathcal{P}$ are continuous functions $D\to\mathbb{R}$ and $D\to I$, respectively, $D$ is a domain with a compact closure.

Recently a deep connection between this operator and different problems from analysis, geometry and even gas dynamic has been discovered. In a series of works some existing and uniqueness problems have been studied as well as the overdeterminedness for some types of the operators $\mathcal{P}$ has been established. Because of the linearity of $\mathcal{P}$ studying homogeneous equation $\mathcal{P}F\approx 0$ and, in particular, searching an approximate solution to this equation provokes the special interest (from both theoretical and practical points of view). It worth noting that even the notion of the approximate solution by itself needs to be defined accurately.

At the first part of the talk I formulate and discuss the new notions identifying problem and approximate solution related to linear functional operator $\mathcal{P}$. In particular, it will be clarified the interrelation of the identifying and well-known Ulam problems. It will be explained also that the latter problem bears a direct relation to the approximate solvability rather then to some mythic stability.

At the second part of the talk the set of linear functional operators for which I succeeded in proving the solvability of the identifying problem and the approximate solvability of the equation $\mathcal{P}F \approx 0$ will be described and discussed.

In conclusion a list of the most interesting unsolved problems will be demonstrated.

**Boris Paneah** *On approximate solvability of the Cauchy equation of arbitrary degree*

The talk is devoted to the well-known but not well studied functional operator

$$\mathfrak{C}_n F := F(0) + \sum_{k=1}^{n} (-1)^k \sum_{1 \leq j_1 < \ldots < j_k \leq n} F(x_{j_1} + \ldots + x_{j_k}),$$

where $x = (x_1, \ldots, x_n)$ is a point of a bounded domain in $\mathbb{R}^n$ and $F$ is a function: $I \to B$ with $B$ a Banach space and $I = \{t : \ 0 \leq t \leq 1\}$. We show at first where from this operator arises in different fields of mathematics and physics, and then we formulate the problem of approximate solvability of the equation $\mathfrak{C}_n F \approx 0$. In the second part of the talk we solve this problem.

**Magdalena Piszczek** *On multivalued iteration semigroups*

Let $K$ be a closed convex cone with a nonempty interior in a Banach space and let $G\colon K \to cc(K)$ be a continuous additive multifunction. The equality

$$F_t \circ G = G \circ F_t, \qquad t \geq 0$$

is a necessary and sufficient condition under which the family $\{F_t, \ t \geq 0\}$ of multifunctions

$$F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x), \qquad x \in K, \ t \geq 0$$

is an iteration semigroup.

**Dorian Popa** *A property of a functional inclusion connected with Hyers-Ulam stability*

We prove that a set-valued map $F\colon X \to \mathcal{P}_0(Y)$ satisfying the functional inclusion $F(x) \Diamond F(y) \subseteq F(x * y)$ admits, in appropriate conditions, a unique selection $f\colon X \to Y$ satisfying the functional equation $f(x) \diamond f(y) = f(x * y)$, where $(X, *)$, $(Y, \diamond)$ are square-symmetric grupoids and $\Diamond$ is the extension of $\diamond$ to the collection $\mathcal{P}_0(Y)$ of all nonempty parts of $Y$.

[1] J. Aczél, *Lectures on functional equations and their applications*, Mathematics in Science and Engineering **19**, Academic Press, New York–London, 1966.

[2] J. Brzdęk, A. Pietrzyk, *A note on stability of the general linear equation*, Aequationes Math. **75** (2008), 267–270.

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[4] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[5] Zs. Páles, *Hyers–Ulam stability of the Cauchy functional equation on square-symmetric grupoids*, Publ. Math. Debrecen **58** (2001), 651–666.

[6] D. Popa, *A stability result for a general linear inclusion*, Nonlinear Funct. Anal. Appl. **9** (2004), 405–414.

[7] D. Popa, *Functional inclusions on square-symmetric grupoids and Hyers–Ulam stability*, Math. Inequal. Appl. **7** (2004), 419–428.

[8] A. Smajdor, *Additive selections of superadditive set-valued functions*, Aequationes Math. **39** (1990), 121–128.

[9] W. Smajdor, *Subadditive set-valued functions*, Glas. Mat. Ser. III **21 (41)** (1986), 343–348.

[10] W. Smajdor, *Superadditive set-valued functions and Banach-Steinhaus theorem*, Rad. Mat. **3** (1987), 203–214.

**Vladimir Yu. Protasov** *Lipschitz stability of linear operators in Banach spaces*

The well-known concept of Ulam–Hyers–Rassias stability for the additive Cauchy equation establishes, in particular, the $p$-stability of linear maps between Banach spaces for all positive parameters $p \neq 1$. The only exception is the Lipschitz case, when $p = 1$ (see [1] and references therein). One of possible ways to obtain stability results for this case is to introduce the notion of Lipschitz linear stability. Let $X, Y$ be arbitrary Banach spaces and $F\colon X \to Y$ be a map with the only assumption that there is $K > 0$ such that $\|F(x)\| \leq K\|x\|$, $x \in X$. For a given $\varepsilon > 0$ we consider the following condition on $F$:

$$\|\{a, b\}_F - \{b, c\}_F\| \leq \varepsilon \qquad a, c \in X,\ b \in [a, c], \tag{1}$$

where $\{x_1, x_2\}_F$ denotes the divided difference $\frac{F(x_2)-F(x_1)}{\|x_2-x_1\|}$. This condition is fulfilled for $\varepsilon = 0$ precisely when $F$ is linear. We say that a map $F$ can be *linearly Lipschitz $C$-approximated* if there is a linear operator $A\colon X \to Y$ such that

$$\|\{x_1, x_2\}_{F-A}\| \leq C, \qquad x_1, x_2 \in X.$$

This means that $\|F(x_1) - F(x_2) - (Ax_1 - Ax_2)\| \leq C\|x_1 - x_2\|$. Observe that if $F(0) = 0$, then $\|F(x) - Ax\| \leq C\|x\|$ for any $x$. Thus, Lipschitz linear approximation property implies the linear approximation in the sense of Ulam–Hyers–Rassias stability for $p = 1$. Consider now the following property called in the sequel Lipschitz linear stability (LLS):

*For given Banach spaces $X$ and $Y$ there is a function $C(\varepsilon)$, which tends to zero as $\varepsilon \to 0$, such that any map $F\colon X \to Y$ possessing property* (1) *can be linearly Lipschitz $C(\varepsilon)$-approximated.*

Any Lipschitz $\varepsilon$-perturbation of a linear operator possesses property (1). The question is whether the converse is true: if (1) holds for a map $F$, then $F$ can be linearly Lipschitz $C(\varepsilon)$-approximated? In other words, if a map $F\colon X \to Y$ can be linearly Lipschitz $\varepsilon$-approximated on any straight line $l \subset X$, can it be $C(\varepsilon)$-approximated globally on $X$? This problem was stated for case of functionals (when $Y = \mathbb{R}$) by Prof. Zsolt Páles in 12th ICFEI [2, Problem 2, pp.150–151] both for the entire space $X$ and for convex domains $D \subset X$. First we answer the question of LLS for functionals:

Theorem 1
*If $X$ is an arbitrary Banach space and $Y = \mathbb{R}$, then the LLS property holds with $C(\varepsilon) = 2\varepsilon$.*

The proof is based on the separation principle, and cannot be extended from the case $Y = \mathbb{R}$ to an arbitrary Banach space $Y$. This extension, nevertheless, can be realized using a totally different idea, which leads to the following result:

Theorem 2
*The LLS property holds with $C(\varepsilon) = 2\varepsilon$ for any Banach spaces $X, Y$, whenever $X$ is separable.*

It appears that the estimate $C(\varepsilon) = 2\varepsilon$ is the best possible in both those theorems, and cannot be improved already for $X = \mathbb{R}^2, Y = \mathbb{R}$. Then we consider LLS for maps $F$ defined on convex open bounded domains $D \subset X$, in which case $C(\varepsilon)$ already depends on the geometry of the domain.

[1] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[2] *Report of Meeting: 12th ICFEI*, Ann. Acad. Pedagog. Crac. Stud. Math. **7** (2008), 125–159.

**Vladimir Yu. Protasov** *Euler binary partition function and refinement equations*

Refinement equations, i.e., difference functional equations with the double contractions of the argument have been studied in the literature in great detail due to their applications in functional analysis, wavelets theory, ergodic theory, probability, etc. Any refinement equation is written in the form

$$\varphi(x) = \sum_{k=0}^{d-1} c_k \varphi(2x - k), \tag{1}$$

where $\{c_k\}$ are complex coefficients such that $\sum_{k=0}^{d-1} c_k = 2$. This equation always possesses a unique, up to multiplication by a constant, compactly supported solution $\varphi$ in the space of distributions $\mathcal{S}'$.

We present a rather surprising application of refinement equations to a well-known problem of the combinatorial number theory: the asymptotics of the Euler partition function. For an arbitrary integer $d \geq 2$ the binary partition function $b(k) = b(d, k)$ is defined on the set of nonnegative integers $k$ as the total number of different binary expansions $k = \sum_{j=0}^{\infty} d_j 2^j$, where the "digits" $d_j$ take values from the set $\{0, \ldots, d-1\}$. The asymptotic behavior of $b(k)$ as $k \to \infty$ was studied by L. Euler, K. Mahler, N.G. de Bruijn, D.E. Knuth, B. Reznick and others.

It appears that the exponent of growth of the function $b(k)$ can be expressed by the solution $\varphi$ of refinement equation (1) with equal coefficients $c_k = \frac{1}{d}$. Using this argument we answer two open questions formulated by B. Reznick in 1990 (see [1]).

[1] B. Reznick, *Some binary partition functions*, Analytic number theory (Allerton Park, IL, 1989), 451–477, Progr. Math. **85**, Birkhäuser Boston, Boston, MA, 1990.

[2] V.Yu. Protasov, *On the problem of the asymptotics of the partition function*, Math. Notes **76** (2004), 144–149.

**Viorel Radu** *Ulam–Hyers stability of functional equations in locally convex probabilistic spaces: a fixed point method*

In [1] and [2] some generalized Ulam–Hyers stability results for Cauchy functional equation have been proved. Our aim is to outline the results concerning the generalized Ulam–Hyers stability for different other kinds of functional equations.

The fixed point method (cf. [4]) will be emphasized, for functions defined on linear spaces and taking values in fuzzy normed spaces and locally convex probabilistic spaces.

[1] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed points approach*, Iteration theory (ECIT'02), 43–52, Grazer Math. Ber. **346**, Karl–Franzens–Univ. Graz, Graz, 2004.

[3] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[4] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

**Ewa Rak** *Domination and distributivity inequalities*
(joint work with **J. Drewniak**)

Domination is a property of operations which plays an important role in considerations connected with the distributivity functional inequalities. Schweizer and Sklar [4] introduced the notion of domination for associative binary operations with common range and common neutral element. In particular, the property of domination was considered in the families of triangular norms and conorms (see e.g. [1, 2, 3]). In our considerations we shall show some of dependencies between the property of domination and the subdistributivity or the superdistributivity of operations on the unit interval.

[1] J. Drewniak, P. Drygaś, U. Dudziak, *Domination between multiplace operations*, Issues in Soft Computing. Decisions and Operations Research, EXIT, Warszawa 2005, 149–160.

[2] S. Saminger-Platz, *The dominance relation in some families of continuous Archimedean t-norms and copulas*, Fuzzy Sets and Systems **160** (2009), 2017–2031.

[3] P. Sarkoci, *Domination in the families of Frank and Hamacher t-norms*, Kybernetica (Prague) **41** (2005), 349–360.

[4] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, 1983.

**Themistocles M. Rassias** *Stanisław Marcin Ulam*

In this special session, I will talk briefly on the life and works of S.M. Ulam.

**Maciej Sablik** *Bisymmetrical functionals*

Let $\Omega_i$, $i = 1, 2$ be compact sets. Consider spaces $B(\Omega_i, \mathbb{R})$ of bounded functions defined on $\Omega_i$, and let $F$ and $G$ be functionals defined in $B(\Omega_1, \mathbb{R})$ and $B(\Omega_2, \mathbb{R})$, respectively. We characterize $F$ and $G$ such that the equation

$$G\left(F_t(x(s,t))\right) = F\left(G_s(x(s,t))\right)$$

holds for every $x \in B(\Omega_1 \times \Omega_2, \mathbb{R})$, under some additional regularity assumptions. It turns out that $F$ and $G$ are conjugated to an integral with respect to some Radon measure in $B_i$. The main tool in the proof is a result of Gy. Maksa from [1].

[1] Gy. Maksa, *Solution of generalized bisymmetry type equations without surjectivity assumptions*, Aequationes Math. **57** (1999), 50–74.

**Ekaterina Shulman** *Stable quasi-mixing of the horocycle flow*
(joint work with **F. Nazarov**)

We consider the behavior of a one-parameter subgroup of a Lie group under the influence of a sequence of kicks. Our approach follows [1] where a special case of the problem was related to an asymptotic behavior of "approximate" solutions of some functional equations on a discrete group.

Let a Lie group $G$ act on a set $X$, and $(h^t)_{t\in\mathbb{R}}$ be a one-parameter subgroup of $G$; it is a dynamical system acting on $X$. We perturb this system by a sequence of kicks $\{\phi_i\} \subset G$. The kicks arrive with some positive period $\tau$. The dynamics of the kicked system is described by a sequence of products $P_\tau(i) = \phi_i h^\tau \phi_{i-1} h^\tau \dots \phi_1 h^\tau$ that depend on the period $\tau$.

A dynamical property of a subgroup $(h^t)$ is called *kick stable*, if for every sequence of kicks $\{\phi_i\}$, the kicked sequence $P_\tau(i)$ inherits this property for a "large" set of periods $\tau$. The property we will concentrate on, is quasi-mixing.

A sequence $\{P(i)\}$ acting on a measure space $(X, \mu)$ by measure-preserving automorphisms is called *quasi-mixing* if there exists a subsequence $\{i_k\} \to \infty$ such that for any two $L_2$-functions $F_1$ and $F_2$ on $X$

$$\int_X F_1(P(i_k)x)F_2(x)\,d\mu \to \int_X F_1(x)\,d\mu \int_X F_2(x)\,d\mu \quad \text{when } k \to \infty.$$

In our case $X = \mathrm{PSL}(2,\mathbb{R})/\Gamma$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a lattice. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on $X$ by left multiplication. The principal tool used in [1] for the study of stable mixing in this setting, is the Howe–Moore theorem which gives the geometric description of quasi-mixing systems: if the sequence $P(i)$ is unbounded then it is quasi-mixing.

It follows from the Howe-Moore theorem that the horocycle flow

$$h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

is quasi-mixing on $X$. We prove that it is kick stably quasi-mixing. This answers the question raised by L. Polterovich and Z. Rudnick in [1].

Let us mention an application to second order difference equations. A discrete Schrödinger-type equation is the equation

$$q_{k+1} - (2 + tc_k)q_k + q_{k-1} = 0, \qquad k \geq 1. \tag{1}$$

COROLLARY.
*For every sequence $\{c_n\}$, the set of the parameters $t \in \mathbb{R}_+$ for which all solutions of the difference equation* (1) *are bounded, has finite measure.*

[1] L. Polterovich, Z. Rudnick, *Kick stability in groups and dynamical systems*, Nonlinearity **14** (2001), 1331–1363.

**Justyna Sikorska** *A direct method for proving the Hyers–Ulam stability of some functional equations*

We study the stability of the equation of the form

$$f(x) = af(h(x)) + bf(-h(x))$$

with some conditions imposed on constants $a, b$ and function $h$. The results are later applied (by use of a direct method – the Hyers sequences) for proving the stability of several functional equations.

**Barbara Sobek** *Quadratic equation of Pexider type on a restricted domain*

Let $X$ be a real (or complex) locally convex linear topological space. Assume that $U$ is a nonempty, open and connected subset of $X \times X$. Let

$$U_1 := \{x : \ (x, y) \in U \text{ for some } y \in X\},$$
$$U_2 := \{y : \ (x, y) \in U \text{ for some } x \in X\}$$

and

$$U_+ := \{x + y : \ (x, y) \in U\},$$
$$U_- := \{x - y : \ (x, y) \in U\}.$$

We consider the functional equation

$$f(x + y) + g(x - y) = h(x) + k(y), \qquad (x, y) \in U,$$

where $f \colon U_+ \to Y$, $g \colon U_- \to Y$, $h \colon U_1 \to Y$ and $k \colon U_2 \to Y$ are unknown functions and $(Y, +)$ is a commutative group. The general solution of the equation is given. We also present an extension result.

**Joanna Szczawińska** *Some remarks on a family of multifunctions*

Let $f\colon \mathbb{R} \to \mathbb{R}$ denote the function given by

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad t \in \mathbb{R},$$

where $a_n \geq 0$ for $n \in \mathbb{N}$. If $K$ is a closed convex cone in a real Banach space and $H\colon K \to \mathrm{cc}(K)$ a linear and continuous set-valued function with nonempty, convex and compact values in $K$, then for all $t \geq 0$ the set-valued function

$$F^t(x) := \sum_{n=0}^{\infty} a_n t^n H^n(x), \qquad x \in K$$

is linear and continuous and

$$F^t \circ F^s(x) \subseteq \sum_{n=0}^{\infty} c_n H^n(x), \qquad x \in K,$$

where

$$c_n = \sum_{k=0}^{n} a_k a_{n-k} t^k s^{n-k}, \qquad t, s \geq 0.$$

The necessary and sufficient condition for the equality

$$F^t \circ F^s(x) = \sum_{n=0}^{\infty} c_n H^n(x), \qquad x \in K,\ t, s \geq 0$$

will be given.

**Tomasz Szostok** *On a functional equation stemming from some property of triangles*

Basing on some geometrical property discovered by G. Monge, in [1] authors considered the following functional equation

$$\left| \frac{1}{2}(y-x) f\left(\frac{x+y}{2}\right) - \frac{1}{2}(f(y) - f(x)) \frac{x+y}{2} \right| = \int_x^y f(t)\, dt + \frac{1}{2} x f(x) - \frac{1}{2} y f(y).$$

They proved that the only solutions of this equation are the affine functions. Roughly speaking this means that Monge theorem works only for collinear points.

In the present talk we modify this equation in such way that it will be satisfied by some functions different from $f(x) = ax + b$. Then we solve the obtained equation.

[1] C. Alsina, M. Sablik, J. Sikorska, *On a functional equation based upon a result of Gaspard Monge*, J. Geom. **85** (2006), 1–6.

**Jacek Tabor** *Approximate $(\varepsilon, p)$-midconvexity for $p \in [0, 1]$*
(joint work with **Józef Tabor** and **M. Żołdak**)

For $p \in [0, 1]$ we put

$$T_p(x) := \sum_{k=0}^{\infty} \frac{1}{2^k} d^p(2^k x), \qquad x \in \mathbb{R},$$

where $d(x) = 2\mathrm{dist}(x, \mathbb{Z})$ and by $0^0$ we understand 0.

A function $f\colon I \to \mathbb{R}$, where $I$ is a subinterval of $\mathbb{R}$, is called *$(\varepsilon, p)$-midconvex* if

$$\mathcal{J}f(x, y) := \frac{f(x) + f(y)}{2} - \frac{f(x) + f(y)}{2} \leq \varepsilon|x - y|^p, \qquad x, y \in I.$$

It is known that if $f$ is a continuous $(\varepsilon, p)$-midconvex function, then

$$f(rx + (1 - r)y) - rf(x) - (1 - r)f(y) \leq \varepsilon T_p(r|x - y|), \qquad x, y \in I,\ r \in [0, 1].$$

The above estimation is optimal for $p = 0$ (theorem of C.T. Ng and K. Nikodem) and $p = 1$ (theorem of Z. Boros). Zs. Palés asked what happens in the case when $p \in [0, 1]$.

We show that the above problem can be reduced to verification of the following hypotheses:

$$\min\{\mathcal{J}d^p(x, y) + \frac{1}{2}d^p(x - y), \mathcal{J}d^p(x, y) + \frac{1}{2}\mathcal{J}d^p(2x, 2y) + \frac{1}{4}d^p(2x - 2y)\} \leq d^p\left(\frac{x - y}{2}\right)$$

for $x, y \in [-1, 1]$. The above inequality can be easily verified for $p = 0$ and $p = 1$ (giving in particular another proof of the result of Z. Boros). Although numerical simulations support the assertion that the above hypothesis holds for all $p \in (0, 1)$, we were not able to prove it.

**Józef Tabor** *Jensen semiconcave functions with power moduli*
(joint work with **Jacek Tabor** and **A. Mureńko**)

We study the relation between Jensen semiconcavity and semiconcavity in the case when modulus of semiconcavity is of the form $\omega(r) = Cr^p$ for $p \in (0, 1]$. As it is known continuous Jensen semiconcave function with modulus $\omega$ is semiconcave with modulus

$$\tilde{\omega}(r) := \sum_{k=0}^{\infty} \omega\left(\frac{r}{2^k}\right).$$

In case of $\omega(r) = Cr^p$ for $p \in (0, 1]$ we improve this result and determine the smallest $\tilde{\omega}$.

**Gheorghe Toader** *Invariance in some families of means*
(joint work with **S. Toader**)

As it is known from the classical example of the arithmetic-geometric mean of Gauss (see [1]), the determination of a $(M, N)-$invariant mean $P$ is a very difficult problem. That is why we study the (equivalent) problem of finding a mean $N$ which is complementary to $M$ with respect to $P$. For the determination of complementaries, three methods have been used: the direct calculation (see [4]), the use of the methods of functional equations (see [2]), and the series expansion of means (see [3]). In the current paper we consider the method of series expansion of means to study the invariance in the family of extended logarithmic means.

[1] J.M. Borwein, P.B. Borwein, *Pi and the AGM. A study in analytic number theory and computational complexity*, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1987.

[2] Z. Daróczy, Zs. Páles, *Gauss-composition of means and the solution of the Matkowski-Sutô problem*, Publ. Math. Debrecen **61** (2002), 157–218.

[3] D.H. Lehmer, *On the compounding of certain means*, J. Math. Anal. Appl. **36** (1971), 183–200.

[4] Gh. Toader, S. Toader, *Greek means and the arithmetic-geometric mean*, RGMIA Monographs, Victoria University, 2005 (http://rgmia.vu.edu.au/monographs).

**Peter Volkmann** *Continuity of solutions of a certain functional equation*

The continuous solutions $f\colon \mathbb{R} \to \mathbb{R}$ of the functional equation

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|$$

had been given in a talk during the Conference on Inequalities and Applications at Noszvaj 2007 (http://riesz.math.klte.hu/∼cia07). Here we show that the continuity of a solution of this functional equation follows from the continuity at one point.

**Marek C. Zdun** *Iteration groups and semigroups – recent results*

This is a survey talk on selected topics concerning iteration groups and semigroups where some progress has been achieved during the last years. Especially we concern on the problem of embeddability of given functions in iteration groups and iterative roots.

In the talk we discuss the following directions in iteration theory:

1. Measurable iteration semigroups.
2. Embedding of diffeomorphisms in regular iteration semigroups on $\mathbb{R}^n$.
3. Iteration groups of fixed point free homeomorphisms on the plane.
4. Embedding of interval homeomorphisms with two fixed points in regular iteration groups.
5. Commuting functions and embeddability.
6. Iterative roots.
7. The structure of iteration groups of homeomorphisms on an interval.
8. The structure of iteration groups of homeomorphisms on the circle.

9. Approximately iterated functions.

10. Set-valued iteration semigroups.

**Marek Żołdak** *Bernstein–Doetsch type theorem for approximately convex functions*
(joint work with **Jacek Tabor** and **Józef Tabor**)

Let $X$ be a real topological vector space, let $D$ be a subset of $X$ and let $\alpha\colon X \to [0,\infty)$ be an even function locally bounded at zero.

A function $f\colon D \to \mathbb{R}$ is called $(\alpha,t)$-*preconvex* (where $t \in (0,1)$ is fixed), if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \alpha(x-y)$$

for all $x, y \in D$ such that $[x,y] \subset D$.

We give a version of Bernstein–Doetsch theorem and some related results for such functions.

## Problems and Remarks

**1. Problem.**

Consider functional equations of the form

$$\sum_{i=1}^{n} a_i f\Big(\sum_{k=1}^{n_i} b_{ik}x_k\Big) = 0, \qquad \sum_{i=1}^{n} a_i \ne 0 \tag{1}$$

and

$$\sum_{i=1}^{m} \alpha_i f\Big(\sum_{k=1}^{m_i} \beta_{ik}x_k\Big) = 0, \qquad \sum_{i=1}^{m} \alpha_i \ne 0, \tag{2}$$

where all parameters are real and $f\colon \mathbb{R} \to \mathbb{R}$.

Assume that the two functional equations are equivalent, i.e., they have the same set of solutions.

Can we say something about the common stability? More precisely, if (1) is stable, what can we say about the stability of (2). Under which additional conditions the stability of (1) implies that of (2)?

*Gian Luigi Forti*

**2. Problem and Remark.**

Let $X$ be a normed space, $D \subseteq X$ be an open convex set and let $f\colon D \to \mathbb{R}$ be a Lipschitz perturbation of a convex function $g\colon D \to \mathbb{R}$, i.e., let $f$ be of the form

$$f = g + \ell,$$

where $g$ is a convex function and $\ell\colon D \to \mathbb{R}$ is $\varepsilon$-Lipschitz, i.e.,

$$|\ell(x) - \ell(y)| \le \varepsilon\|x-y\|, \qquad x, y \in D.$$

Then, for $x, y \in D$ and $t \in [0,1]$, we have

$$\begin{aligned} f(tx+(1-t)y) &- tf(x) - (1-t)f(y) \\ &= \big[g(tx+(1-t)y) - tg(x) - (1-t)g(y)\big] \\ &\quad + \big[\ell(tx+(1-t)y) - t\ell(x) - (1-t)\ell(y)\big] \\ &\le t\big[\ell(tx+(1-t)y) - \ell(x)\big] + (1-t)\big[\ell(tx+(1-t)y) - \ell(y)\big] \\ &\le t\big|\ell(tx+(1-t)y) - \ell(x)\big| + (1-t)\big|\ell(tx+(1-t)y) - \ell(y)\big| \\ &\le t\varepsilon\|(tx+(1-t)y) - x\| + (1-t)\varepsilon\|(tx+(1-t)y) - y\| \\ &= 2\varepsilon t(1-t)\|x-y\|. \end{aligned}$$

Therefore, $f$ satisfies the approximate convexity inequality:

$$f(tx+(1-t)y) \le tf(x) + (1-t)f(y) + 2\varepsilon t(1-t)\|x-y\|. \tag{1}$$

On the other hand, in the case $X = \mathbb{R}$, we have the following converse of the above observation (which is a particular case of a result obtained in [1]).

Proposition.
*Let $I$ be an open interval and $\varepsilon \ge 0$. Assume that $f\colon I \to \mathbb{R}$ satisfies, for all $x, y \in I$ and $t \in [0,1]$, inequality (1). Then there exists a convex function $g : I \to \mathbb{R}$ such that the function $\ell := f - g$ is $(2\varepsilon)$-Lipschitz.*

The following more general and open problem seems to be of interest.

Problem
Does there exist a constant $\gamma$ (that may depend on $X$ and $D$) such that, whenever a function $f\colon D \to X$ satisfies inequality (1) for all $x, y \in D$ and $t \in [0,1]$, then there exists a convex function $g\colon D \to \mathbb{R}$ such that the function $\ell := f - g$ is $\gamma\varepsilon$-Lipschitz on $D$?

A result related to this problem was stated by V. Protasov during the 13th ICFEI:

*If a function $f\colon X \to \mathbb{R}$ satisfies, for all $x, y \in X$ and $t \in [0,1]$,*

$$\big|f(tx+(1-t)y) - tf(x) - (1-t)f(y)\big| \le 2\varepsilon t(1-t)\|x-y\|,$$

*then there exists a continuous linear functional $x^* \in X^*$ such that $\ell := f - x^*$ is $(4\varepsilon)$-Lipschitz on $X$.*

[1] Zs. Páles, *On approximately convex functions*, Proc. Amer. Math. Soc. **131** (2003), 243–252.

*Zsolt Páles*

**3. Problem.**
Let $X$ be a Hilbert space, $D \subseteq X$ an open convex set, $\varepsilon > 0$ and let $f\colon D \to \mathbb{R}$ be a continuous function such that

$$f(tx+(1-t)y) - tf(x) - (1-t)f(y) \le \varepsilon t(1-t)\|x-y\|, \quad x, y \in D,\ t \in [0,1].$$

Does there exist an $x_0 \in D$ such that $f$ is differentiable at $x_0$?

This problem is motivated by the results of S. Rolewicz.

*Jacek Tabor and Józef Tabor*

**4. Problem.**

In connection with some problem in theoretical physics, O.G. Bokov introduced in [1] the following functional equation

$$f(x,y)f(x+y,z)+f(y,z)f(y+z,x)+f(z,x)f(z+x,y)=0. \tag{1}$$

In [2] A.V. Yagzhev determined all analytic solutions $f\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ of (1). However, his proof is not clear and presents several gaps. So, we may wonder about the validity of the result. Therefore, the problem is to find all analytic solutions $f\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of (1) with a nice mahematical proof. Also, we may ask about the solutions of (1) in a more general setting.

[1] O.G. Bokov, *A model of Lie fields and multiple-time retarded Green's functions of an electromagnetic field in dielectric media*, Nauchn. Tr. Novosib. Gos. Pedagog. Inst. **86** (1973), 3–9.

[2] A.V. Yagzhev, *A functional equation from theoretical physics*, Funct. Anal. Appl. **16** (1982), 38–44.

*Nicole Brillouët-Belluot*

**5. Remark.**

During the last fifteen years a great number of papers concerning stability of functional equations have been published. Unfortunately in many of these papers motivations for studying a given equation or/and possible applications of the stability results are missing. In my opinion this will eventually produce a discredit of the topic and, consequently, a discredit of the field of functional equations: a thing that we, functional equationists, certainly do not want. These considerations are mainly directed to younger colleagues, in order to invite them to investigate genuine, not rather artificial, mathematical problems.

*Gian Luigi Forti*

**6. Remark.**

Let $(X, \|\cdot\|)$ be a normed space, $D \subset X$ be a convex set and $c > 0$ be a fixed constant. A function $f\colon D \to \mathbb{R}$ is called *strongly convex with modulus $c$* if

$$f\big(tx+(1-t)y\big) \le tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2 \tag{1}$$

for all $x, y \in D$ and $t \in [0,1]$. Under the assumption (A) that $(X, \|\cdot\|)$ is an inner product space, the following equivalence (B) holds:

*$f\colon D \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if $g = f - c\|\cdot\|^2$ is convex.*

The following example gives an answer to the question posed by Zsolt Páles after my talk at this conference and shows that assumption (A) is essential for (B).

Example.
Let $X = \mathbb{R}^2$ and $\|x\| = |x_1| + |x_2|$ for $x = (x_1, x_2)$. Take $f = \|\cdot\|^2$. Then $g = f - \|\cdot\|^2 = 0$ is convex. However, $f$ is not strongly convex with modulus 1. Indeed, for $x = (1,0)$ and $y = (0,1)$ we have

$$f\Big(\frac{x+y}{2}\Big) = 1 > 0 = \frac{f(x)+f(y)}{2} - \frac{1}{4}\|x-y\|^2,$$

which contradicts (1).

One can also prove that if (B) holds for every $f\colon X \to \mathbb{R}$, then $(X, \|\cdot\|)$ must be an inner product space. Thus condition (B) gives another characterization of the inner product spaces among normed spaces.

*Kazimierz Nikodem*

**7. Remark.**

The Institute of Mathematics of the Pedagogical University of Cracow accepted in 1983 for realization Dobiesław Brydak's proposal of continuing in Poland the series of five international conferences on functional equations, which had been organized by our Hungarian colleagues at Miskolc and Debrecen from 1966 to 1979 (see [1]).

The First International Conference on Functional Equations and Inequalities was held at Sielpia in Kielce region of Poland from May 27 to June 2, 1984. In fact, it was a second conference on functional equations held in Poland, ever after that organized by Professors Stanisław Gołąb and Marek Kuczma at Zakopane in October 9-13, 1967 (see [2]). The organizers of the 1st ICFEI were Dobiesław Brydak, Bogdan Choczewski and Józef Tabor. The meeting was opened (and then attended) by Professor Zenon Moszner, Rector Magnificus of the Pedagogical University of Cracow (see [3]).

The general statistical data, concerning 1st, 13th and all ICFEIs (in brackets: the numbers of different persons participating) are presented in Table 1, whereas in Tables 2 and 3 the distribution of participants into countries and cities (of affiliation) is exhibited. Table 4 shows the number of all ICFEIs the participant of the 13th one attended, with "$\star$" meaning her or his presence at the 1st ICFEI. (All the data have been collected by Miss Janina Wiercioch, a member of organizing staffs from 1991 (3rd ICFEI) on.)

| ICFEI | All participants | Foreign participants | Countries | Talks | Sessions |
|---|---|---|---|---|---|
| 1st | 59 | 9 | 8 | 41 | 8 |
| 13th | 76 | 31 | 10 | 73 | 26 |
| All 13 | 857 (269) | 206 (111) | 32 | 694 | 239 |

Table 1. General data

| Country | 1st ICFEI | 13th ICFEI | Cities |
|---|---|---|---|
| Australia | 2 | - | La Trobe, Melbourne |
| Austria | 1 | 1 | Graz ‖ Innsbruck |
| Czechoslovakia | 1 | - | Brno |
| France | - | 1 | Nantes |
| Germany | - | 2 | Clausthal-Zellerfeld, Lobau |
| Greece | - | 1 | Athens |
| Hungary | 2 | 14 | Miskolc ‖ Debrecen 13, Miskolc 1 |
| Israel | - | 1 | Haifa |
| Italy | 1 | 1 | Milan ‖ Milan |
| Romania | - | 5 | Cluj-Napoca 3, Timişoara 2 |
| Russia | - | 5 | Moscow 1, Nizhny Novgorod 3, Vologda 1 |
| Switzerland | 1 | - | Bern |
| West Germany | 1 | - | Karlsruhe |
| $\sum$ | 9 | 31 | |

Table 2. Participants from abroad

| City | 1st ICFEI | 13th ICFEI |
|---|---|---|
| Białystok | 1 | - |
| Bielsko-Biała | 3 | 3 |
| Częstochowa | 1 | - |
| Gdańsk | 1 | 1 |
| Gliwice | - | 2 |
| Katowice | 11 | 12 |
| Kielce | 4 | 1 |
| Kraków | 22 | 16 |
| Rzeszów | 6 | 8 |
| Zielona Góra | - | 2 |
| $\sum$ | 50 | 45 |

Table 3. Polish participants

According to Table 4 in the 13th ICFEI took part 13 colleagues who also attended our first meeting held 25 years ago. Among them were: Karol Baron, Roman Ger, Maciej Sablik (all from Katowice) who participated in all ICFEIs, and from abroad: Gian Luigi Forti (Milan) and Peter Volkmann (Karlsruhe) who took part in 5, respectively 10, conferences. Moreover, what may be surprising, at the 13th ICFEI were present less Polish mathematicians than in the 1st one. One can also observe that 9 colleagues (7 from abroad) came to our conference for the first time (at least four of them seemed to be younger than the ICFEI).

The most numerous group of our guests from abroad usually was that of Hungarians (altogether 85 presences, 14 participants of the 13th ICFEI). The author then proposed to transform the popular saying on Hungarian-Polish fraternity as follows:

*Magyar-Lengyel jó barát - igen függvényegyenletek, igen függvényegyenlötlenségek* (Hungarian and Pole are good nephews - both in functional equations and inequalities).

| | | | |
|---|---|---|---|
| R. Badora | 9 | J. Mako | 1 |
| A. Bahyrycz | 5 | G. Maksa | 6 |
| Sz. Baják | 1 | F. Mészáros | 3 |
| K. Baron | 13 ⋆ | B. Micherda | 1 |
| L. Bartłomiejczyk | 8 | V. Mityushev | 3 |
| S. Belmesova | 1 | L. Molnár | 3 |
| M. Bessenyei | 2 | J. Morawiec | 10 |
| Z. Boros | 6 | J. Mrowiec | 5 |
| N. Brillouët-Belluot | 7 | A. Mureńko | 5 |
| J. Brzdęk | 9 | A. Najdecki | 5 |
| P. Burai | 2 | K. Nikodem | 10 ⋆ |
| L. Cădariu | 2 | A. Nuyatov | 1 |
| J. Chmieliński | 11 | A. Olbryś | 3 |
| B. Choczewski | 12 ⋆ | J. Olko | 6 |
| J. Chudziak | 7 | B. Paneah | 4 |
| K. Ciepliński | 6 | Z. Páles | 9 |
| M. Czerni | 11 ⋆ | M. Piszczek | 4 |
| S. Czerwik | 9 | V.D. Popa | 2 |
| Z. Daróczy | 8 | V.Yu. Protasov | 2 |
| J. Dascal | 2 | B. Przebieracz | 3 |
| J. Domsta | 8 | V. Radu | 2 |
| A. Filchenkov | 1 | E. Rak | 2 |
| G.-L. Forti | 5 ⋆ | Th.M. Rassias | 3 |
| W. Förg-Rob | 10 | M. Sablik | 13 ⋆ |
| R. Ger | 13 ⋆ | E. Shulman | 3 |
| A. Gilányi | 6 | J. Sikorska | 8 |
| D. Głazowska | 4 | A. Smajdor | 10 ⋆ |
| E. Gselmann | 1 | B. Sobek | 2 |
| G. Guzik | 6 | P. Solarz | 6 |
| A. Házy | 3 | J. Szczawińska | 10 |
| E. Jabłońska | 3 | T. Szostok | 6 |
| H.-H. Kairies | 11 | Jacek Tabor | 7 |
| B. Koclęga-Kulpa | 4 | Józef Tabor | 12 ⋆ |
| Z. Kominek | 12 ⋆ | G. Toader | 4 |
| D. Krassowska | 5 | S. Toader | 1 |
| Z. Leśniak | 9 | P. Volkmann | 10 ⋆ |
| A. Mach | 8 ⋆ | M.C. Zdun | 10 ⋆ |
| E. Mainka | 1 | M. Żołdak | 4 |

Table 4. Numbers of all ICFEIs attended by participants

[1] B. Choczewski, *International meetings organized by Polish schools on functional equations*, Ann. Acad. Pedagog. Crac. Stud. Math. **5** (2006), 13–32.

[2] *Międzynarodowa Konferencja z Równań Funkcyjnych, International Conference on Functional Equations, Zakopane, 9.X.-13.X.1967*, Zeszyty Nauk Uniw. Jagiello. Prace Mat. **14** (1970).

[3] *Proceedings of the International Conference on Functional Equations and Inequalities, May 27 - June 2, 1984, Sielpia (Poland)*, Rocznik Nauk.-Dydakt. Prace Mat. **11** (1985), 185–265.

*Bogdan Choczewski*

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