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A note on the convergence of partial Szász-Mirakyan type operators

Abstract. In this paper we study approximation properties of partial modified Szasz-Mirakyan operators for functions from exponential weight spaces. We present some direct theorems giving the degree of approximation for these operators. The considered version of Szász-Mirakyan operators is more useful from the computational point of view.

1. Introduction

Let us denote by $C(\mathbb{R}_0)$ a set of all real-valued functions continuous on $\mathbb{R}_0 = [0; +\infty)$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In paper [1] we investigated operators of Szász-Mirakyan type defined as follows

$$A_n^\nu(f;x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n}\Big), & x > 0; \\ f(0), & x = 0 \end{cases}$$

and

$$p_{n,k}^\nu(x) = \frac{1}{I_\nu(nx)} \frac{x^{2k+\nu}}{2^{2k+\nu} k!\, \Gamma(k+\nu+1)},$$

where Γ is the gamma function and I_ν stands for the modified Bessel function, i.e.

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{2k+\nu}}{2^{2k+\nu} k!\, \Gamma(k+\nu+1)}.$$

We studied the operators in polynomial weight spaces

$$C_p = \{f \in C(\mathbb{R}_0) : \ w_p f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

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where w_p was the polynomial weight function defined as follows

$$w_p(x) = \begin{cases} 1, & p = 0; \\ \dfrac{1}{1+x^p}, & p \in \mathbb{N} \end{cases}$$

for $x \in \mathbb{R}_0$. The space C_p is a normed space with the norm

$$\|f\|_p = \sup_{x \in \mathbb{R}_0} w_p(x)|f(x)|.$$

In paper [4] a certain modification of the above operators was introduced

$$B_n^\nu(f, a_n; x) = \begin{cases} \displaystyle\sum_{k=0}^{[n(x+a_n)]} p_{n,k}^\nu(x) f\Big(\frac{2k}{n}\Big), & x > 0; \\ f(0), & x = 0 \end{cases}$$

for $f \in C_p$, where (a_n) was a sequence of positive numbers such that

$$\lim_{n\to\infty} a_n \sqrt{n} = \infty \tag{1}$$

and $[n(x+a_n)]$ denotes the integral part of $n(x+a_n)$.

In the paper were studied approximation properties of these operators. Among others there was deduced that

$$\lim_{n\to\infty} \{B_n^\nu(f, a_n; x) - f(x)\} = 0$$

for every $f \in C_p$, uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. In that case the crucial assumption was that the sequence of positive numbers (a_n) satisfied the condition (1).

Similar problems for Baskakov type operators were discussed in paper [5] and some generalization of truncated operators we can find in [6].

We shall drive analogous results for the modified Szász-Mirakyan operators for functions from exponential weight spaces

$$E_q = \{f \in C(\mathbb{R}_0) : \ v_q f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where v_q is the exponential weight function defined as follows

$$v_q(x) = e^{-qx}, \qquad q \in \mathbb{R}_+$$

for $x \in \mathbb{R}_0$. The space E_q is a normed space with the norm

$$\|f\|_q = \sup_{x \in \mathbb{R}_0} v_q(x)|f(x)|. \tag{2}$$

In this paper we will present a certain modification of the operator B_n^ν. We shall apply the modification to prove the convergence of the operators in exponential weight spaces.

2. Main results

At the beginning of this section we will recall the definition of $\overline{A}_n^\nu$ and some preliminary results from papers [2] and [3], which we shall apply to prove main theorems.

If $\nu \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we considered operators of Szász-Mirakyan type as follows

$$\overline{A}_n^\nu(f;x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big), & x > 0; \\ f(0), & x = 0 \end{cases}$$

for $f \in E_q$.

Lemma 1
For all $\nu \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$ we have

$$\overline{A}_n^\nu(1;x) = 1.$$

Lemma 2
For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for each $n \in \mathbb{N}$ we have

$$\Big\|\overline{A}_n^\nu\Big(\frac{1}{v_q};\cdot\Big)\Big\|_q \leq M(q,\nu).$$

Applying the definition of $\overline{A}_n^\nu$ and (2) we get

Lemma 3
For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for each $n \in \mathbb{N}$ we have

$$\big\|\overline{A}_n^\nu(f;\cdot)\big\|_q \leq M(q,\nu)\|f\|_q.$$

Notice that operators $\overline{A}_n^\nu$ are bounded and transform the space E_q into itself.

Lemma 4
For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for each $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have

$$\big|\overline{A}_n^\nu((t-x)^2;x)\big| \leq M(q,\nu)\frac{x(x+1)}{n}.$$

Lemma 5
For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have

$$v_q(x)\overline{A}_n^\nu\Big(\frac{(t-x)^2}{v_q(t)};x\Big) \leq M(q,\nu)\frac{x(x+1)}{n}.$$

Theorem 6
If $\nu \in \mathbb{R}_0$ *and* $f \in E_q$ *with some* $q \in \mathbb{R}_+$*, then for all* $x \in \mathbb{R}_0$

$$\lim_{n\to\infty} \left\{\overline{A}_n^{\nu}(f;x) - f(x)\right\} = 0.$$

Moreover, the above convergence is uniform on every set $[x_1, x_2]$ *with* $0 \le x_1 < x_2$.

Proof. Let $f \in E_q$ with some $q \in \mathbb{R}_+$. Pick $x \in \mathbb{R}_0$ and $\varepsilon > 0$. There exists a number δ such that

$$|f(t) - f(x)| < \frac{\varepsilon}{2} \tag{3}$$

for $|t - x| < \delta$, $t \in \mathbb{R}_0$. By linearity of $\overline{A}_n^{\nu}$ and Lemma 1 we get

$$\begin{aligned}
&\left|\overline{A}_n^{\nu}(f;x) - f(x)\right| \\
&\quad \le \overline{A}_n^{\nu}(|f - f(x)|;x) \\
&\quad = \sum_{|\frac{2k}{n+q}-x|<\delta} p_{n,k}^{\nu}(x)\left|f\Big(\frac{2k}{n+q}\Big) - f(x)\right| + \sum_{|\frac{2k}{n+q}-x|\ge\delta} p_{n,k}^{\nu}(x)\left|f\Big(\frac{2k}{n+q}\Big) - f(x)\right| \\
&\quad = I_1 + I_2.
\end{aligned}$$

Hence by (3) we obtain $I_1 < \frac{\varepsilon}{2}$. Further we get

$$I_2 \le \frac{\|f\|_q}{\delta}\,\overline{A}_n^{\nu}\Big(\frac{|t-x|}{v_q(t)};x\Big) + \frac{\|f\|_q}{\delta v_q(x)}\,\overline{A}_n^{\nu}(|t-x|;x).$$

Using the Hölder inequality and Lemmas 2–5 we have

$$\begin{aligned}
I_2 &\le \frac{\|f\|_q}{\delta}\left[\overline{A}_n^{\nu}\Big(\frac{(t-x)^2}{v_q(t)};x\Big)\right]^{\frac12}\left[\overline{A}_n^{\nu}\Big(\frac{1}{v_q(t)};x\Big)\right]^{\frac12} + \frac{\|f\|_q}{\delta v_q(x)}\left[\overline{A}_n^{\nu}((t-x)^2;x)\right]^{\frac12} \\
&\le \frac{\|f\|_q}{\delta}\left[M(q,\nu)\frac{x(x+1)}{n v_q(x)}\right]^{\frac12}\left[M(q,\nu)\frac{1}{v_q(x)}\right]^{\frac12} + \frac{\|f\|_q}{\delta v_q(x)}\left[M(q,\nu)\frac{x(x+1)}{n}\right]^{\frac12} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

The above estimations imply the convergence in Theorem 6.

In the space E_q we define the following class of partial operators

$$\overline{B}_n^{\nu}(f,a_n;x) = \begin{cases} \displaystyle\sum_{k=0}^{[(n+q)(x+a_n)]} p_{n,k}^{\nu}(x) f\Big(\frac{2k}{n+q}\Big), & x > 0; \\ f(0), & x = 0, \end{cases}$$

where we replace the infinite summing in $\overline{A}_n^{\nu}$ by the finite one and we still have the assumption (1).

Theorem 7
If $\nu \in \mathbb{R}_0$ *and* $f \in E_q$ *with some* $q \in \mathbb{R}_+$ *then for all* $x \in \mathbb{R}_0$

$$\lim_{n\to\infty} \left\{\overline{B}_n^{\nu}(f,a_n;x) - f(x)\right\} = 0.$$

Moreover, the above convergence is uniform on every interval $[x_1, x_2]$*,* $x_2 > x_1 \ge 0$.

Proof. Let $f \in E_q$ with some $q \in \mathbb{R}_+$. From the definitions of the operators $\overline{A}_n^\nu$ and $\overline{B}_n^\nu$ we get

$$\begin{aligned}
&\overline{B}_n^\nu(f, a_n; x) - f(x) \\
&\quad = \sum_{k=0}^{[(n+q)(x+a_n)]} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big) - f(x) \\
&\quad = \sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big) - f(x) - \sum_{k=[(n+q)(x+a_n)]+1}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big) \\
&\quad = \overline{A}_n^\nu(f; x) - f(x) - R_n^\nu(f, a_n; x)
\end{aligned}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Observe that

$$\Big|\frac{2k}{n+q} - x\Big| \geq a_n \qquad \text{if} \qquad k \geq [(n+q)(x+a_n)] + 1.$$

Analogously as in the previous proof we can write the following estimation

$$\begin{aligned}
\big|R_n^\nu(f, a_n; x)\big| &\leq \sum_{k=[(n+q)(x+a_n)]+1}^{\infty} p_{n,k}^\nu(x) \Big| f\Big(\frac{2k}{n+q}\Big)\Big| \\
&\leq \sum_{|\frac{2k}{n+q} - x| \geq a_n} p_{n,k}^\nu(x) \Big| f\Big(\frac{2k}{n+q}\Big)\Big| \leq \frac{\|f\|_q}{a_n} \sum_{k=0}^{\infty} p_{n,k}^\nu(x) \frac{|\frac{2k}{n+q} - x|}{v_q(\frac{2k}{n+q})} \\
&= \frac{\|f\|_q}{a_n} \overline{A}_n^\nu\Big(\frac{|t-x|}{v_q(t)}; x\Big).
\end{aligned}$$

The Hölder inequality, Lemmas 2 and 5 imply

$$\begin{aligned}
\big|R_n^\nu(f, a_n; x)\big| &\leq \frac{\|f\|_q}{a_n} \Big[\overline{A}_n^\nu\Big(\frac{(t-x)^2}{v_q(t)}; x\Big)\Big]^{\frac{1}{2}} \Big[\overline{A}_n^\nu\Big(\frac{1}{v_q(t)}; x\Big)\Big]^{\frac{1}{2}} \\
&\leq M(q, \nu) \frac{\|f\|_q}{a_n} \frac{\sqrt{x(x+1)}}{\sqrt{n}} \frac{1}{v_q(x)}.
\end{aligned}$$

In view of (1) we obtain the required result.

Notice that the same simplified method we can use in polynomial weight spaces C_p to estimate the reminder of the series A_n^ν, which was considered in [4].

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