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Naga Vijay Krishna Dasari, Jakub Kabat

Several observations about Maneeals - a peculiar system of lines

Abstract. For an arbitrary triangle ABC and an integer n we define points D_n, E_n, F_n on the sides BC, CA, AB respectively, in such a manner that

$$\frac{|AC|^n}{|AB|^n} = \frac{|CD_n|}{|BD_n|}, \qquad \frac{|AB|^n}{|BC|^n} = \frac{|AE_n|}{|CE_n|}, \qquad \frac{|BC|^n}{|AC|^n} = \frac{|BF_n|}{|AF_n|}.$$

Cevians AD_n, BE_n, CF_n are said to be the *Maneeals of order n*. In this paper we discuss some properties of the Maneeals and related objects.

1. Introduction

Given an arbitrary triangle ABC we consider certain cevians [1] defined (for given integral n) in the following way.

Definition 1.1
Let ABC be a triangle and let n be an integer. There are points D_n, E_n, F_n on the sides BC, CA, AB respectively, satisfying

$$\frac{|AC|^n}{|AB|^n} = \frac{|CD_n|}{|BD_n|}, \qquad \frac{|AB|^n}{|BC|^n} = \frac{|AE_n|}{|CE_n|}, \qquad \frac{|BC|^n}{|AC|^n} = \frac{|BF_n|}{|AF_n|}.$$

We call the cevians AD_n, BE_n, CF_n the *order n Maneeals* of the triangle ABC.

It is easy to check, that medians, bisectors and symmedians of a triangle are examples of the Maneeals with integer $n = 0$, $n = 1$, $n = 2$ respectively. Furthermore, for the given triangle ABC, the points D_n, E_n, F_n (uniquely determined by the integer n), are vertices of a new triangle, which is said to be the *Maneeal's*

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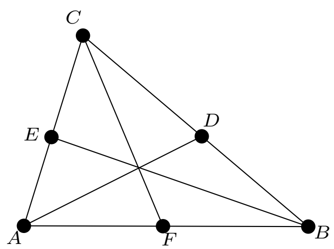
triangle of order n. For Maneeals AD_n, BE_n, CF_n we can use the Ceva's theorem in order to prove that they are concurrent. Their point of intersection M_n is said to be the *Maneeal's point of order n.* For given M_n we can choose points P_n, Q_n, R_n on the sides BC, CA, AB respectively, in such a manner that line segments M_nP_n, M_nQ_n, M_nR_n are perpendicular to the corresponding sides of the triangle ABC. Points P_n, Q_n, R_n are the vertices of a next triangle [11], which is said to be the *Maneeal's pedal triangle of order n.*

In the present note we investigate properties of Maneeal's points and pedal triangles.

In this paper, for a given triangle ABC and an integer n the points D_n, E_n, F_n, P_n, Q_n, R_n should be always understood in accordance with the definitions given above. Furthermore, we will use the following notation:
$|AB| = c,\ \ |BC| = a,\ \ |CA| = b,\ \ q_n = a^n + b^n + c^n,$
Δ_{XYZ} - the area of triangle XYZ,
$\Delta = \Delta_{ABC},\ \ \Delta_n = \Delta_{D_nE_nF_n},\ \ \Delta'_n = \Delta_{P_nQ_nR_n},$
R - the radius of the circumcircle of triangle ABC,
S - circumcenter of triangle ABC.

2. First properties

THEOREM 2.1 (Ceva's Theorem [4, 5])
For a given triangle with vertices A, B, and C and non-collinear points D, E, and F on lines BC, AC, AB respectively, a necessary and sufficient condition for the cevians AD, BE, and CF to be concurrent (intersect in a single point) is that $|BD| \cdot |CE| \cdot |AF| = |DC| \cdot |EA| \cdot |FB|$.



THEOREM 2.2 (Maneeal's points)
Maneeals of order n are always concurrent.

Proof. Since the points D_n, E_n, F_n lie always between the vertices A, B, C, they cannot be collinear. The equality

$$\frac{|BD_n|}{|CD_n|} \cdot \frac{|CE_n|}{|AE_n|} \cdot \frac{|AF_n|}{|BF_n|} = \frac{b^n}{c^n} \cdot \frac{c^n}{a^n} \cdot \frac{a^n}{c^n} = 1$$

together with Theorem (2.1) proves the assertion.

In the next Lemma we collect a number of properties connected with Maneeals and related objects.

Lemma 2.3
The lenghts of the segments of Maneeal's triangle of order n are given by the following formulas:

$$|E_nF_n|^2 = \frac{b^{2n}c^2(a^n+c^n)^2 + b^2c^{2n}(a^n+b^n)^2 - b^nc^n(a^n+b^n)(a^n+c^n)(b^2+c^2-a^2)}{(a^n+b^n)^2(a^n+c^n)^2},$$

$$|E_nD_n|^2 = \frac{a^{2n}b^2(c^n+b^n)^2 + a^2b^{2n}(c^n+a^n)^2 - a^nb^n(c^n+a^n)(c^n+b^n)(a^2+b^2-c^2)}{(c^n+b^n)^2(c^n+a^n)^2},$$

$$|F_nD_n|^2 = \frac{a^{2n}c^2(b^n+c^n)^2 + a^2c^{2n}(b^n+a^n)^2 - a^nc^n(b^n+a^n)(b^n+c^n)(a^2+c^2-b^2)}{(b^n+c^n)^2(b^n+a^n)^2}.$$

Lemma 2.4
The lenghts of the segments in which the Manneals of order n divide the sides of the triangle are given by the following formulas:

$$|BD_n| = \frac{c^n \cdot a}{b^n+c^n}, \qquad |CD_n| = \frac{b^n \cdot a}{b^n+c^n},$$
$$|CE_n| = \frac{a^n \cdot b}{c^n+a^n}, \qquad |AE_n| = \frac{c^n \cdot b}{c^n+a^n},$$
$$|AF_n| = \frac{b^n \cdot c}{b^n+a^n}, \qquad |BF_n| = \frac{a^n \cdot c}{b^n+a^n}.$$

Lemma 2.5
The Manneal's point M_n of order n divides the Manneal's segments of order n in the following ratios:

$$\frac{|AM_n|}{|M_nD_n|} = \frac{c^n+b^n}{a^n}, \qquad \frac{|BM_n|}{|M_nE_n|} = \frac{c^n+a^n}{b^n}, \qquad \frac{|CM_n|}{|M_nF_n|} = \frac{a^n+b^n}{c^n}.$$

Lemma 2.6
It is also convenient to keep, for further reference, record of the following identities:

$$\begin{aligned} \frac{|AM_n|}{|AD_n|} &= \frac{c^n+b^n}{q_n}, & \frac{|M_nD_n|}{|AD_n|} &= \frac{a^n}{q_n}, \\ \frac{|BM_n|}{|BE_n|} &= \frac{c^n+a^n}{q_n}, & \frac{|M_nE_n|}{|BE_n|} &= \frac{b^n}{q_n}, \\ \frac{|CM_n|}{|CF_n|} &= \frac{a^n+b^n}{q_n}, & \frac{|M_nF_n|}{|CF_n|} &= \frac{c^n}{q_n}, \end{aligned} \tag{1}$$

and

$$|AD_n|^2 = \frac{b^2c^2}{(b^n+c^n)^2}[(b^n+c^n)(b^{n-2}+c^{n-2}) - a^2b^{n-2}c^{n-2}],$$
$$|BE_n|^2 = \frac{a^2c^2}{(a^n+c^n)^2}[(a^n+c^n)(a^{n-2}+c^{n-2}) - b^2a^{n-2}c^{n-2}], \tag{2}$$
$$|CF_n|^2 = \frac{b^2a^2}{(b^n+a^n)^2}[(b^n+a^n)(b^{n-2}+a^{n-2}) - c^2b^{n-2}a^{n-2}].$$

Lemma 2.7

The Maneeal's point of order n and every pair of vertices of the given triangle ABC determine new triangles. Areas of these triangles are related to the areas of the triangle ABC in the following way:

$$\Delta_{BM_nC} = \frac{a^n}{q_n}\cdot\Delta, \qquad \Delta_{CM_nA} = \frac{b^n}{q_n}\cdot\Delta, \qquad \Delta_{BM_nA} = \frac{c^n}{q_n}\cdot\Delta. \tag{3}$$

We can also consider the triangles determined by pairs of points D_n, E_n, F_n and one of the vertices of the triangle ABC. Their areas are given by the following formulas:

$$\Delta_{BD_nF_n} = \frac{a^nc^n}{(b^n+c^n)(b^n+a^n)}\cdot\Delta,$$
$$\Delta_{CD_nE_n} = \frac{a^nb^n}{(c^n+b^n)(c^n+a^n)}\cdot\Delta, \tag{4}$$
$$\Delta_{AE_nF_n} = \frac{b^nc^n}{(a^n+c^n)(a^n+b^n)}\cdot\Delta.$$

Lemma 2.8

It is also worth to note some properties connected with the lengths of line segments between the vertices of pedal Maneeals triangles of order n, the Maneeals points of order n, and vertices of the triangle ABC.

$$|M_nP_n| = \frac{2\Delta a^{n-1}}{q_n}, \qquad |M_nQ_n| = \frac{2\Delta b^{n-1}}{q_n}, \qquad |M_nR_n| = \frac{2\Delta c^{n-1}}{q_n}. \tag{5}$$

$$|BP_n| = \frac{1}{q_n}\cdot\sqrt{a^2c^2(a^n+c^n)(a^{n-2}+c^{n-2}) - b^2a^nc^n - 4a^{2n-2}\Delta^2},$$
$$|BR_n| = \frac{1}{q_n}\cdot\sqrt{a^2c^2(a^n+c^n)(a^{n-2}+c^{n-2}) - b^2a^nc^n - 4c^{2n-2}\Delta^2},$$
$$|CP_n| = \frac{1}{q_n}\cdot\sqrt{a^2b^2(a^n+b^n)(a^{n-2}+b^{n-2}) - c^2a^nb^n - 4a^{2n-2}\Delta^2},$$
$$|CQ_n| = \frac{1}{q_n}\cdot\sqrt{a^2b^2(a^n+b^n)(a^{n-2}+b^{n-2}) - c^2a^nb^n - 4b^{2n-2}\Delta^2}, \tag{6}$$
$$|AQ_n| = \frac{1}{q_n}\cdot\sqrt{c^2b^2(c^n+b^n)(c^{n-2}+b^{n-2}) - a^2c^nb^n - 4b^{2n-2}\Delta^2},$$
$$|AR_n| = \frac{1}{q_n}\cdot\sqrt{c^2b^2(c^n+b^n)(c^{n-2}+b^{n-2}) - a^2c^nb^n - 4c^{2n-2}\Delta^2}.$$

Corollary 2.9
By definition we have $|BP_n|+|CP_n|=a$, $|CQ_n|+|AQ_n|=b$, $|BR_n|+|AR_n|=c$. Hence, by adding all equations in (6)*, we get*

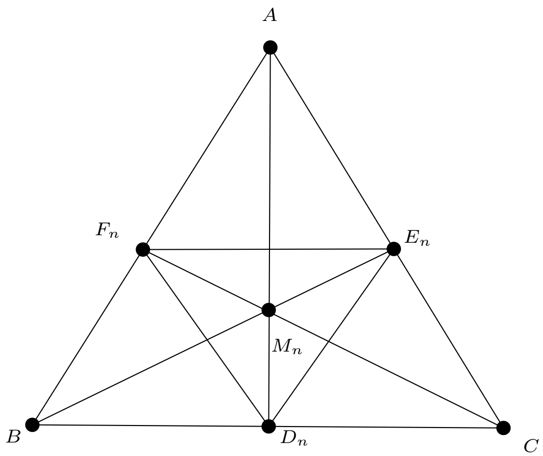
$$\begin{aligned}&(a+b+c)\cdot(a^n+b^n+c^n)\\&\quad=\sqrt{a^2c^2(a^n+c^n)(a^{n-2}+c^{n-2})-b^2a^nc^n-4a^{2n-2}\Delta^2}\\&\qquad+\sqrt{a^2c^2(a^n+c^n)(a^{n-2}+c^{n-2})-b^2a^nc^n-4c^{2n-2}\Delta^2}\\&\qquad+\sqrt{a^2b^2(a^n+b^n)(a^{n-2}+b^{n-2})-c^2a^nb^n-4a^{2n-2}\Delta^2}\\&\qquad+\sqrt{a^2b^2(a^n+b^n)(a^{n-2}+b^{n-2})-c^2a^nb^n-4b^{2n-2}\Delta^2}\\&\qquad+\sqrt{c^2b^2(c^n+b^n)(c^{n-2}+b^{n-2})-a^2c^nb^n-4b^{2n-2}\Delta^2}\\&\qquad+\sqrt{c^2b^2(c^n+b^n)(c^{n-2}+b^{n-2})-a^2c^nb^n-4c^{2n-2}\Delta^2}.\end{aligned}$$

Now we are in a position to prove the following new identity.

Theorem 2.10

$$\begin{aligned}\Delta_n&=2\cdot\frac{a^nb^nc^n}{(a^n+b^n)(b^n+c^n)(c^n+a^n)}\cdot\Delta,\\ \Delta_{-n}&=\Delta_n.\end{aligned}$$

Proof. We will start with simple consequences of the definition of Maneeals:



$$|AE_n|=\frac{c^n}{a^n}|E_nC|,\qquad |BF_n|=\frac{a^n}{b^n}|F_nA|.$$

From Lemma 2.4 we obtain

$$\frac{|AE_n|}{|AC|}=\frac{c^n}{c^n+a^n},\qquad \frac{|BF_n|}{|BA|}=\frac{a^n}{a^n+b^n}.$$

On the other hand we can observe, that

$$\frac{|AE_n|}{|AC|} = \frac{\Delta ABE_n}{\Delta} \qquad \text{and} \qquad \frac{|BF_n|}{|BA|} = \frac{\Delta_{BE_nF_n}}{\Delta_{BE_nA}}.$$

Finally we can express the area of the triangle BE_nF_n by the formula

$$\begin{aligned}\Delta_{BE_nF_n} &= \frac{|BF_n|}{|BA|} \cdot \Delta_{BE_nA} = \frac{a^n}{a^n+b^n} \cdot \Delta_{BE_nA} \\ &= \frac{a^n}{a^n+b^n} \cdot \frac{|AE_n|}{|AC|} \cdot \Delta = \frac{a^n}{a^n+b^n} \cdot \frac{c^n}{c^n+a^n} \cdot \Delta \\ &= \frac{a^nc^n}{(a^n+b^n)(c^n+a^n)} \cdot \Delta.\end{aligned}$$

Strictly analogously, we can provide the following formulas:

$$\begin{aligned}\Delta_{BE_nD_n} &= \frac{c^na^n}{(c^n+b^n)(a^n+c^n)} \cdot \Delta, \\ \Delta_{BF_nD_n} &= \frac{a^nc^n}{(a^n+b^n)(c^n+b^n)} \cdot \Delta.\end{aligned}$$

To end the first part of the theorem, we only need to note, that

$$\Delta_n = \Delta_{BE_nD_n} + \Delta_{BE_nF_n} - \Delta_{BF_nD_n}.$$

Finally we get

$$\begin{aligned}\Delta_n &= \frac{c^na^n}{(c^n+b^n)(a^n+c^n)} \cdot \Delta + \frac{a^nc^n}{(a^n+b^n)(c^n+a^n)} \cdot \Delta - \frac{a^nc^n}{(a^n+b^n)(c^n+b^n)} \cdot \Delta \\ &= \frac{a^nc^n}{(a^n+b^n)(b^n+c^n)(c^n+a^n)}[(a^n+b^n)+(b^n+c^n)-(c^n+a^n)] \cdot \Delta.\end{aligned}$$

Above consideration implies that $\Delta_n = 2 \cdot \frac{a^nb^nc^n}{(a^n+b^n)(b^n+c^n)(c^n+a^n)} \cdot \Delta$.

The last part of the proof is quite formal.

$$\begin{aligned}\Delta_{-n} &= 2 \cdot \frac{a^{-n}b^{-n}c^{-n}}{(a^{-n}+b^{-n})(b^{-n}+c^{-n})(c^{-n}+a^{-n})} \cdot \Delta \\ &= 2 \cdot \frac{a^{-n}b^{-n}c^{-n}}{(a^{-n}+b^{-n})(b^{-n}+c^{-n})(c^{-n}+a^{-n})} \cdot \Delta \cdot \frac{a^{2n}b^{2n}c^{2n}}{a^{2n}b^{2n}c^{2n}} \\ &= 2 \cdot \frac{a^nb^nc^n}{(a^n+b^n)(b^n+c^n)(c^n+a^n)} \cdot \Delta \\ &= \Delta_n.\end{aligned}$$

Now we pass to the Maneeal's pedal triangle $P_nQ_nR_n$.

Lemma 2.11
The area of Maneeals pedal triangle of order n is given by the following formula

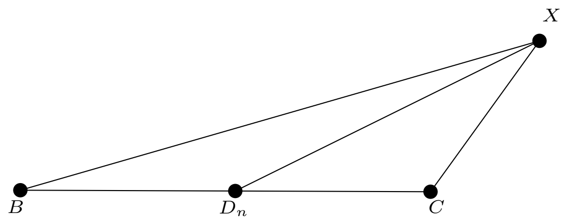
$$\Delta'_n = \frac{2^{2n-2}\Delta^{n+1}R^{n-2}}{(a^n+b^n+c^n)^2}[a^{2-n}+b^{2-n}+c^{2-n}].$$

Furthermore, sides of Maneeals pedal triangle have the following lengths:

$$|P_nQ_n| = \frac{2\Delta}{q_n}\sqrt{(a^{n-2}+b^{n-2})(a^n+b^n)-a^{n-2}b^{n-2}c^2},$$
$$|Q_nR_n| = \frac{2\Delta}{q_n}\sqrt{(b^{n-2}+c^{n-2})(b^n+c^n)-b^{n-2}c^{n-2}a^2},$$
$$|R_nP_n| = \frac{2\Delta}{q_n}\sqrt{(c^{n-2}+a^{n-2})(c^n+a^n)-c^{n-2}a^{n-2}b^2}.$$

Theorem 2.12
For any point X in the plane we have the following formula

$$|M_nX|^2 = \frac{a^n|AX|^2+b^n|BX|^2+c^n|CX|^2}{a^n+b^n+c^n} - \frac{a^2b^2c^2}{(a^n+b^n+c^n)^2}(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}).$$



Proof. From the Stewart Theorem [3] for triangle XBC, we get

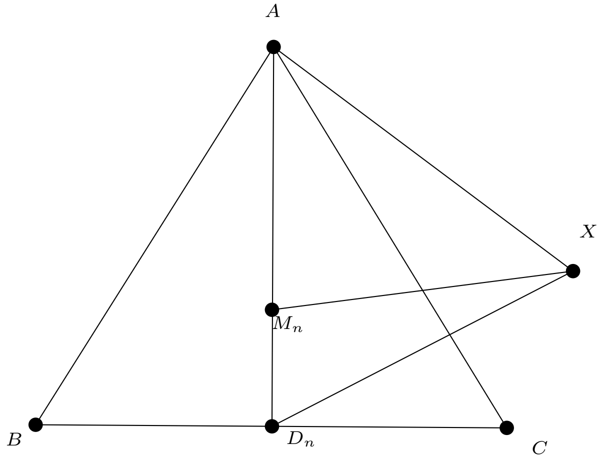
$$|XB|^2\cdot|D_nC|+|XC|^2\cdot|D_nB| = |BC|\cdot[|XD_n|^2+|D_nC|\cdot|D_nB|],$$

or equivalently

$$|XB|^2\cdot\frac{|D_nC|}{|BC|}+|XC|^2\cdot\frac{|D_nB|}{|BC|} = |XD_n|^2+|D_nC|\cdot|D_nB|.$$

If we use formulas from Lemma 2.4 and make a few simple transformations, then we will get

$$|XD_n|^2 = \frac{c^n}{c^n+b^n}|XC|^2+\frac{b^n}{c^n+b^n}|XB|^2-\frac{b^nc^n}{(c^n+b^n)^2}|BC|^2.$$



Furthermore, an analogous consideration for the triangle AXD_n shows that

$$\begin{aligned}
|XM_n|^2 &= \frac{|AM_n|}{|AD_n|}|D_nX|^2 + \frac{|D_nM_n|}{|AD_n|}|AX|^2 - |AM_n|\cdot|D_nM_n| \\
&= \frac{|AM_n|}{|AD_n|}|D_nX|^2 + \frac{|D_nM_n|}{|AD_n|}|AX|^2 - \frac{|AM_n|}{|AD_n|}\cdot\frac{|D_nM_n|}{|AD_n|}\cdot|AD_n|^2 \\
&= \Big(\frac{c^n+b^n}{a^n+b^n+c^n}\Big)\Big(\frac{c^n}{c^n+b^n}|XC|^2 + \frac{b^n}{c^n+b^n}|XB|^2 - \frac{b^nc^na^2}{(c^n+b^n)^2}\Big) \\
&\quad + \frac{a^n}{a^n+b^n+c^n}|AX|^2 - \frac{a^n(c^n+b^n)}{(a^n+b^n+c^n)^2}|AD_n|^2 \\
&= \frac{a^n|AX|^2+b^n|BX|^2+c^n|CX|^2}{a^n+b^n+c^n} - \frac{a^2b^nc^n}{(a^n+b^n+c^n)(b^n+c^n)} \\
&\quad - \frac{a^n(c^n+b^n)}{(a^n+b^n+c^n)^2}|AD_n|^2.
\end{aligned}$$

Now we will use formula

$$|AD_n|^2 = \frac{b^2c^2}{(b^n+c^n)^2}[(b^n+c^n)(b^{n-2}+c^{n-2}) - a^2b^{n-2}c^{n-2}]$$

to simplify the following part of the main formula

$$\begin{aligned}
&-\frac{a^2b^nc^n}{(a^n+b^n+c^n)(b^n+c^n)} - \frac{a^n(c^n+b^n)}{(a^n+b^n+c^n)^2}|AD_n|^2 \\
&\quad = -\frac{a^2b^nc^n}{(a^n+b^n+c^n)(b^n+c^n)} \\
&\qquad - \frac{a^n(c^n+b^n)}{(a^n+b^n+c^n)^2}\frac{b^2c^2}{(b^n+c^n)^2}[(b^n+c^n)(b^{n-2}+c^{n-2}) - a^2b^{n-2}c^{n-2}]
\end{aligned}$$

$$= -\frac{a^2b^2c^2}{(a^n+b^n+c^n)^2}[a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}].$$

Finally we have

$$\begin{aligned}|M_nX|^2 = &\frac{a^n|AX|^2+b^n|BX|^2+c^n|CX|^2}{a^n+b^n+c^n}\\ &-\frac{a^2b^2c^2}{(a^n+b^n+c^n)^2}(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}).\end{aligned} \tag{7}$$

Corollaries 2.13
From (7) *we obtain for* $n = 0, 1$*:*

$$|M_0X|^2 = \frac{|AX|^2+|BX|^2+|CX|^2}{3} - \frac{a^2+b^2+c^2}{9},$$
$$|M_1X|^2 = \frac{a|AX|^2+b|BX|^2+c|CX|^2}{a+b+c} - \frac{abc}{(a+b+c)}.$$

Note that M_0 *is the centroid of the triangle* ABC *and* M_1 *is the incenter of* ABC*.*

From (7), (1), (2) *we obtain the last corollary, which states, that the distance between two Maneeal's points, of order* m *and* n*, is given by the following formula*

$$|M_mM_n|^2 = \frac{1}{(a^m+b^m+c^m)^2(a^n+b^n+c^n)^2}\cdot V,$$

where

$$\begin{aligned}V = &a^2\{[b^mc^n-b^nc^m]^2-[b^mc^n-b^nc^m][a^m(b^n-c^n)-a^n(b^m-c^m)]\\ &-[a^mb^n-a^nb^m][a^mc^n-a^nc^m]\}\\ &+b^2\{[c^ma^n-c^na^m]^2-[c^ma^n-c^na^m][b^m(c^n-a^n)-b^n(c^m-a^m)]\\ &-[b^mc^n-b^nc^m][b^ma^n-b^na^m]\}\\ &+c^2\{[a^mb^n-a^nb^m]^2-[a^mb^n-a^nb^m][c^m(a^n-b^n)-c^n(a^m-b^m)]\\ &-[c^ma^n-c^na^m][c^mb^n-c^nb^m]\}.\end{aligned}$$

In particular if we let $m = 1$*,* $n = 0$*, we will get*

$$|M_1M_0|^2 = \frac{1}{(a+b+c)}[a|AM_0|^2+b|BM_0|^2+c|CM_0|^2-abc].$$

Corollary 2.14
From (2.12) *for X=S we have*

$$\begin{aligned}|M_nS|^2 = &\frac{a^nR^2+b^nR^2+c^nR^2}{a^n+b^n+c^n}\\ &-\frac{a^2b^2c^2}{(a^n+b^n+c^n)^2}(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2})\\ &\geq 0.\end{aligned}$$

Therefore, we have

$$R^2 \geq \frac{a^2b^nc^n + b^2a^nc^n + c^2a^nb^n}{q_n^2}.$$

In particular, for $n = 1$, since

$$R^2 \geq \frac{abc}{(a+b+c)} = 2Rr$$

therefore

$$R \geq 2r.$$

Other proofs of Euler's inequality you can find at [2, 13, 14, 6, 7, 8, 9, 10].

Now we can obtain a few relationships from the *Cauchy-Schwarz inequality*. They will be important part of proofs of several subsequent theorems.

Lemma 2.15
Any non-zero real numbers a, b, c satisfy the following inequalities:

$$\left(\frac{a^{2n}}{a^2} + \frac{b^{2n}}{b^2} + \frac{c^{2n}}{c^2}\right) \geq \frac{(a^n + b^n + c^n)^2}{a^2 + b^2 + c^2}, \tag{8}$$

$$(a^{2n} + b^{2n} + c^{2n}) \geq \frac{1}{3}(a^n + b^n + c^n)^2, \tag{9}$$

$$\left(\frac{a^2}{a^n} + \frac{b^2}{b^n} + \frac{c^2}{c^n}\right) \geq \frac{(a+b+c)^2}{a^n + b^n + c^n}. \tag{10}$$

Proof. Indeed, these are special cases of the Cauchy-Schwarz inequality

$$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \geq (x_1y_1 + x_2y_2 + x_3y_3)^2$$

with the substitutions

$$x_1 = \frac{a^n}{a},\ x_2 = \frac{b^n}{b},\ x_3 = \frac{c^n}{c},\ y_1 = a,\ y_2 = b,\ y_3 = c \qquad \text{for (8)}$$

$$x_1 = a^n,\ x_2 = b^n,\ x_3 = c^n,\ y_1 = y_2 = y_3 = 1 \qquad \text{for (9)}$$

and

$$x_1 = \sqrt{\frac{a^2}{a^n}},\ x_2 = \sqrt{\frac{b^2}{b^n}},\ x_3 = \sqrt{\frac{c^2}{c^n}},\ y_1 = \sqrt{a^n},\ y_2 = \sqrt{b^n},\ y_3 = \sqrt{c^n} \qquad \text{for (10).}$$

Lemma 2.16 (Inequality of arithmetic and geometric means)
For any real numbers $x_1, \ldots, x_n$ there is

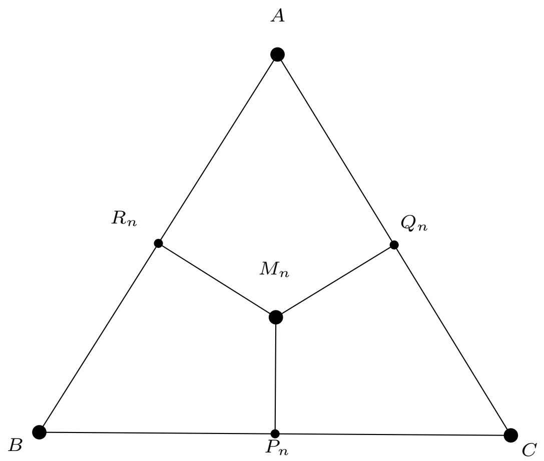
$$\frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \cdot \ldots \cdot x_n}. \tag{11}$$

The Symmedian point M_2 has a special feature, which we will describe by following

Theorem 2.17
Let $S(n) = |M_nP_n|^2 + |M_nQ_n|^2 + |M_nR_n|^2$. Then $S(2) \leq S(n)$ for all $n \in \mathbb{Z}$.

Proof. Using (5) we get

$$S(n) = \frac{4\Delta^2}{q_n^2}\left[\frac{a^{2n}}{a^2} + \frac{b^{2n}}{b^2} + \frac{c^{2n}}{c^2}\right].$$



Using (8) we get

$$S(n) \geq \frac{4\Delta^2}{q_2}.$$

Now, an easy computation shows, that

$$S(2) = |M_2P_2|^2 + |M_2Q_2|^2 + |M_2R_2|^2 = \frac{4\Delta^2}{q_2}.$$

This finishes the proof.

Not only the Symmedian point, but also the Centroid M_0, has a special feature:

Theorem 2.18
Let $T(n) = a^2|M_nP_n|^2 + b^2|M_nQ_n|^2 + c^2|M_nR_n|^2$. Then $T(0) \leq T(n)$ for all $n \in \mathbb{Z}$.

Proof. Using (5) we obtain

$$T(n) = 4\Delta^2 \cdot \frac{q_{2n}}{q_n^2}.$$

By (9) we get

$$T(n) \geq \frac{4\Delta^2}{3}.$$

Now an easy computation shows, that

$$a^2|M_0P_0|^2 + b^2|M_0Q_0|^2 + c^2|M_0R_0|^2 = \frac{4\Delta^2}{3}.$$

This finishes the proof.

Theorem 2.19
Let $W(n) = \frac{a}{|M_nP_n|} + \frac{b}{|M_nQ_n|} + \frac{c}{|M_nR_n|}$. *Then* $W(1) \leq S(n)$ *for all* $n \in \mathbb{Z}$.

Proof. Using (5) we get

$$W(n) = \frac{q_n}{2\Delta}\Big(\frac{a^2}{a^n} + \frac{b^2}{b^n} + \frac{c^2}{c^n}\Big).$$

Using (10) we get

$$W(n) = \frac{(a^n + b^n + c^n)}{2\Delta}\Big(\frac{a^2}{a^n} + \frac{b^2}{b^n} + \frac{c^2}{c^n}\Big) \geq \frac{(a+b+c)^2}{2\Delta}.$$

Now an easy computation shows, that

$$\frac{a}{|M_1P_1|} + \frac{b}{|M_1Q_1|} + \frac{c}{|M_1R_1|} = \frac{(a+b+c)^2}{2\Delta}.$$

This finishes the proof.

Theorem 2.20
Let $K(n) = |M_nP_n| \cdot |M_nQ_n| \cdot |M_nR_n|$. *Then* $K(0) \geq K(n)$ *for all* $n \in \mathbb{Z}$.

Proof. Using (5) we obtain

$$K(n) = \frac{8a^{n-1}b^{n-1}c^{n-1}}{(a^n + b^n + c^n)^3} \cdot \Delta^3.$$

By a special case of 2.16 we get

$$(a^n + b^n + c^n) \geq 3\sqrt[3]{a^nb^nc^n}.$$

Equivalently,

$$\frac{1}{(a^n + b^n + c^n)^3} \leq \frac{1}{27a^nb^nc^n}.$$

Using this inequality, we get

$$K(n) \leq \frac{8a^{n-1}b^{n-1}c^{n-1}}{27a^n b^n c^n} \cdot \Delta^3 = \frac{8}{27abc} \cdot \Delta^3.$$

Furthermore,

$$(a^n + b^n + c^n) = 3\sqrt[3]{a^n b^n c^n} \qquad \text{for } n = 0.$$

Hence,

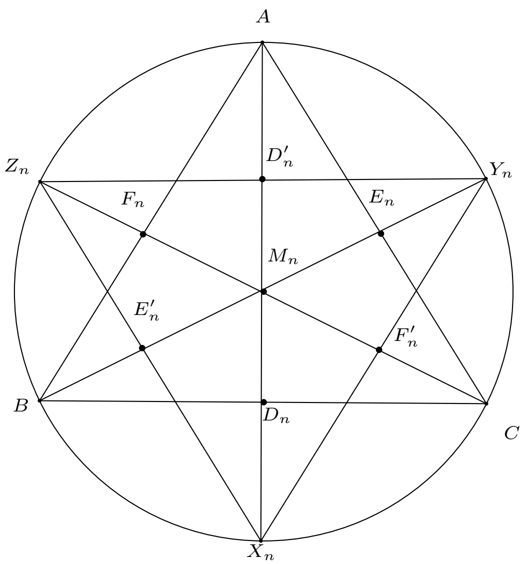
$$|M_0P_0| \cdot |M_0Q_0| \cdot |M_0R_0| = \frac{8a^{-1}b^{-1}c^{-1}}{27} \cdot \Delta^3 = \frac{8}{27abc} \cdot \Delta^3.$$

This equality finishes the proof.

Theorem 2.21
Let π be the circumcircle of the triangle ABC. For any $n \in \mathbb{Z}$ we choose points X_n, Y_n, Z_n on π in such a manner, that chords AX_n, BY_n, CZ_n contain the Maneeals AD_n, BE_n, CF_n respectively. Let D'_n, E'_n, F'_n be intersection points of AX_n, BY_n, CZ_n and Y_nZ_n, Z_nX_n, X_nY_n respctively.

Let finally m be an integer, $X_nD''_m$, $Y_nE''_m$, $Z_nF''_m$ be order m Maneeals of the triangle $X_nY_nZ_n$. Then the following conditions hold:

- $D''_2 = D'_2$, $E''_2 = E'_2$, $F''_2 = F'_2$,
- *if G_m is an m-order Maneeal's point of triangle $X_nY_nZ_n$, then $G_2 = M_2$.*



Proof. Line segments BC and AX_n are the chords of circle π, and D_n is their point of intersection. Hence,

$$|BD_n| \cdot |D_nC| = |AD_n| \cdot |D_nX_n|.$$

Using 2.4 we get

$$|D_nX_n| = \frac{a^2b^nc^n}{(b^n+c^n)^2|AD_n|}.$$

Strictly analogously we get

$$|E_nY_n| = \frac{b^2a^nc^n}{(a^n+c^n)^2|BE_n|}, \qquad |F_nZ_n| = \frac{c^2a^nb^n}{(a^n+b^n)^2|CF_n|}.$$

Now we can use (1) and (2) to obtain

$$\begin{aligned}|M_nX_n| = |M_nD_n| + |D_nX_n| &= \frac{a^n|AD_n|}{(a^n+b^n+c^n)} + \frac{a^2b^nc^n}{(b^n+c^n)^2|AD_n|} \\ &= \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{(a^n+b^n+c^n)(b^n+c^n)|AD_n|}.\end{aligned}$$

Strictly analogously we get

$$\begin{aligned}|M_nY_n| &= \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{(a^n+b^n+c^n)(c^n+a^n)|BE_n|}, \\ |M_nZ_n| &= \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{(a^n+b^n+c^n)(a^n+b^n)|CF_n|}.\end{aligned}$$

Using (2) again, we get

$$|AX_n| = |AD_n| + |D_nX_n| = \frac{|AD_n|^2}{|AD_n|} + \frac{a^2b^nc^n}{(b^n+c^n)^2|AD_n|} = \frac{c^2b^n + b^2c^n}{(b^n+c^n)|AD_n|}.$$

Analogously we get

$$|BY_n| = \frac{c^2a^n + a^2c^n}{(a^n+c^n)|BE_n|}, \qquad |CZ_n| = \frac{b^2a^n + a^2b^n}{(a^n+b^n)|CF_n|}.$$

Now we observe, that triangles $X_nM_nZ_n$ and CM_nA are similar. Indeed, we only need to note, that M_n is the intersection point of chords AX_n and CZ_n and that respective angles are right.

From the similarity, we derive that

$$\frac{\Delta X_nM_nZ_n}{\Delta CM_nA} = \frac{|X_nM_n|^2}{|CM_n|^2} = \frac{|X_nZ_n|^2}{|CA|^2} = \frac{|Z_nM_n|^2}{|AM_n|^2}.$$

Since $\frac{|X_nZ_n|}{|CA|} = \frac{|Z_nM_n|}{|AM_n|}$, therefore if we use (1), we get

$$|X_nZ_n| = \frac{|AC| \cdot |Z_nM_n|}{|AM_n|} = b \cdot \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{(a^n+b^n)(b^n+c^n)|AD_n||CF_n|}.$$

Strictly analogously we get

$$|X_nY_n| = c \cdot \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{(a^n + c^n)(b^n + c^n)|AD_n||BE_n|},$$
$$|Y_nZ_n| = a \cdot \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{(a^n + c^n)(a^n + b^n)|CF_n||BE_n|}.$$

By (3) and similarity of respective triangles we have

$$\begin{aligned}\Delta X_nM_nZ_n &= \frac{|X_nZ_n|^2}{|CA|^2} \cdot \Delta CM_nA \\ &= \frac{b^n(a^2b^nc^n + b^2c^na^n + c^2a^nb^n)^2}{(a^n + b^n + c^n)(a^n + b^n)^2(b^n + c^n)^2|AD_n|^2|CF_n|^2} \cdot \Delta.\end{aligned}$$

By the same taken we get

$$\Delta Y_nM_nZ_n = \frac{a^n(a^2b^nc^n + b^2c^na^n + c^2a^nb^n)^2}{(a^n + b^n + c^n)(a^n + b^n)^2(a^n + c^n)^2|BE_n|^2|CF_n|^2} \cdot \Delta,$$
$$\Delta X_nM_nY_n = \frac{c^n(a^2b^nc^n + b^2c^na^n + c^2a^nb^n)^2}{(a^n + b^n + c^n)(c^n + b^n)^2(c^n + a^n)^2|AD_n|^2|BE_n|^2} \cdot \Delta.$$

No we can determine the area of the triangle Z_nBY_n. Since

$$\frac{\Delta_{Z_nM_nY_n}}{\Delta_{Z_nBY_n}} = \frac{|Y_nM_n|}{|BY_n|},$$

therefore

$$\Delta_{Z_nBY_n} = \frac{BY_n}{Y_nM_n} \cdot \Delta_{Z_nM_nY_n} = \frac{a^n(a^2b^nc^n + b^2c^na^n + c^2a^nb^n)(c^2a^n + a^2c^n)}{(a^n + b^n)^2(a^n + c^n)^2|BE_n|^2|CF_n|^2} \cdot \Delta.$$

Strictly analogously we get

$$\Delta_{X_nBY_n} = \frac{c^n(a^2b^nc^n + b^2c^na^n + c^2a^nb^n)(a^nc^2 + c^na^2)}{(c^n + b^n)^2(c^n + a^n)^2|BE_n|^2|AD_n|^2} \cdot \Delta.$$

Furthermore we have the following relation

$$\frac{|Z_nE'_n|}{|X_nE'_n|} = \frac{\Delta_{Z_nBY_n}}{\Delta_{X_nBY_n}} = \frac{a^n(b^n + c^n)^2|AD_n|^2}{c^n(b^n + a^n)^2|CF_n|^2}.$$

On the other hand since

$$\frac{|Y_nZ_n|}{|Y_nX_n|} = \frac{a(b^n + c^n)|AD_n|}{c(b^n + a^n)|CF_n|},$$

we can conclude, that

$$\frac{|Y_nZ_n|^2}{|Y_nX_n|^2} = \frac{|Z_nE'_n|}{|X_nE'_n|} \quad \text{if and only if} \quad n = 2.$$

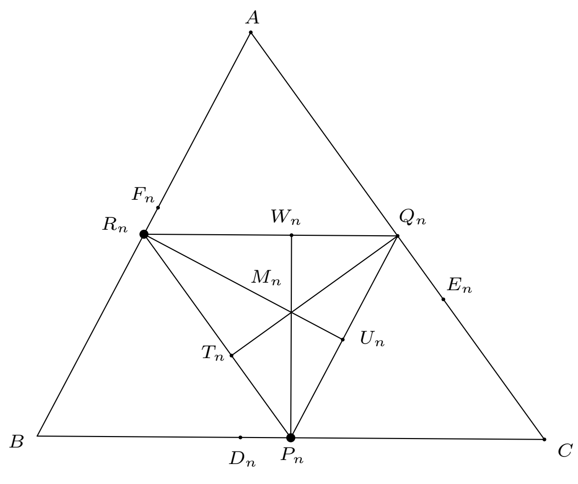
On the other hand, by definition, the cevian $Y_nE''_m$, is an order m Maneeal of triangle $X_nY_nZ_n$ if, and only if, the following condition holds

$$\frac{|Z_nE''_m|}{|X_nE''_m|} = \frac{|Y_nZ_n|^m}{|Y_nX_n|^m}.$$

Now it is easy to see, that the above condition is satisfied for $n = m = 2$. Hence, $E''_2 = E'_2$. We can provide strictly analogously, that $D''_2 = D'_2$, $F''_2 = F'_2$. In particular, cevians $X_2D'_2$, $Y_2E'_2$, $Z_2F'_2$ are Symmedians of the triangle $X_2Y_2Z_2$. The fact, that $G_2 = M_2$ we conclude by the definition (construction) of points X_2, Y_2, Z_2.

Theorem 2.22 (Lemoine's Pedal Triangle Theorem [15])
Let $D_nE_nF_n$ be the order n Maneeal's triangle and $P_nQ_nR_n$ be the order n pedal Maneeal's triangle of the given triangle ABC. We can choose points T_n, U_n, W_n, on sides R_nP_n, Q_nP_n, Q_nR_n, respectively, in such a manner that cevians R_nU_n, P_nW_n, Q_nT_n contain the line segments R_nM_n, P_nM_n, Q_nM_n respectively. Then the following conditions hold:

- *if $R_nR'_m$, $Q_nQ'_m$, $P_nP'_m$ are order m Maneeals of the triangle $P_nQ_nR_n$, then $U_2 = R'_0$, $W_2 = P'_0$, $T_2 = Q'_0$,*
- *Symmedian point of triangle ABC is the centroid of its pedals triangle $P_nQ_nR_n$,*
- $\frac{|AF_1|}{|F_1B|} = \frac{|P_1U_1|}{|U_1Q_1|}$, $\frac{|BD_1|}{|D_1C|} = \frac{|Q_1W_1|}{|W_1R_1|}$, $\frac{|CE_1|}{|E_1A|} = \frac{|R_1T_1|}{|T_1P_1|}$.



Proof. We use (5) to make simple computations

$$\frac{|P_nU_n|}{|U_nQ_n|} = \frac{\Delta_{R_nM_nP_n}}{\Delta_{R_nM_nQ_n}} = \frac{\frac{1}{2}|R_nM_n||M_nP_n|\sin(\angle R_nM_nP_n)}{\frac{1}{2}|R_nM_n||M_nQ_n|\sin(\angle R_nM_nQ_n)}$$

$$= \frac{\frac{2c^{n-1}\Delta}{(a^n+b^n+c^n)} \frac{2a^{n-1}\Delta}{(a^n+b^n+c^n)} \sin(\angle 180 - B)}{\frac{2c^{n-1}\Delta}{(a^n+b^n+c^n)} \frac{2b^{n-1}\Delta}{(a^n+b^n+c^n)} \sin(\angle 180 - A)} = \frac{a^{n-1}\sin B}{b^{n-1}\sin A}$$
$$= \frac{a^{n-2}}{b^{n-2}}$$

and get

$$\frac{|P_nU_n|}{|U_nQ_n|} = \frac{a^{n-2}}{b^{n-2}}. \tag{12}$$

In particular, if we choose $n = 2$, we obtain $|P_nU_n| = |U_nQ_n|$. Therefore, by definition of order m Maneeals of the triangle $P_nQ_nR_n$ the cevian R_2U_2 is a median of this triangle. Finally $R_2U_2 = R_2R_0'$. Strictly analogously we can show, that $P_2W_2 = P_2P_0'$, $Q_2T_2 = Q_2Q_0'$.

On the other hand, if we put $n = 1$ into (12), we obtain

$$\frac{|P_1U_1|}{|U_1Q_1|} = \frac{b}{a} = \frac{|AF_1|}{|F_1B|}.$$

And analogously

$$\frac{|Q_1W_1|}{|W_1R_1|} = \frac{c}{b} = \frac{|BD_1|}{|D_1C|}, \qquad , \frac{|R_1T_1|}{|T_1P_1|} = \frac{a}{c} = \frac{|CE_1|}{|E_1A|}.$$

For further properties see [12].

Final remarks. In this article we introduced Maneeal's points, which to the best of our knowledge, have non been studied in the literature before. We shared a number of properties of these points, related lines and triangles. There is surely much more to discover. We hope, this note will sparkle some interest in the construction and will lead further research in this area of very classical triangle geometry.

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Naga Vijay Krishna Dasari
Department of Mathematics
Kesava Reddy Educational Instutions
Machilipatnam, Kurnool, Andhra Pradesh
India
E-mail: vijay9290009015@gmail.com

Jakub Kabat
Institute of Mathematics
Pedagogical University of Cracow
Podchorazych 2
30-084 Kraków
Poland
E-mail: xxkabat@gmail.com

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