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Line arrangements in algebraic terms

(doctoral thesis)

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Chapter 1

Introduction

The theory of hyperplane arrangements is one of the most classical theories in combinatorial algebraic geometry. The origins of the theory date back to the ancient Greece (4th century), for instance the celebrated Pappus theorem which leads to the so-called symmetric (9_3)-configuration. Another interesting classical result which involves a point-line configuration is Desargues's theorem (17th century): if two triangles are in central perspective (lines joining pairs of their vertices intersect in one point), then the triangles are in axial perspective (intersection points of pairs of lines containing edges of the triangles are collinear). In these results the most important common factor is the *collinearity* of certain collections of points. Years later, in 1893, Sylvester asked if there exists an arrangement $\mathcal{L}$ in the real projective plane which is not a pencil of lines, such that there are no double intersection points. A baby case of this problem was raised already in 1821 by Jackson and is known as the *orchard problem*. Somewhat surprisingly, this problem was solved to the negative by Gallai (or Grünwald) around 1940.

Theorem 1.0.1 (Sylvester-Gallai)**.** *If $\mathcal{L}$ is an arrangement of d lines in the real projective plane which does not contain any double intersection point, then all lines have a common point of intersection, i.e. they form a pencil of d concurrent lines.*

In 1941, Melchior presented his inequality for line arrangements in the real projective plane that gives another proof of Sylvester's problem [30]. For an arrangement of lines $\mathcal{L}$ in a projective plane we denote by t_r the number of r-fold points, i.e., points where exactly r lines from $\mathcal{L}$ meet.

Theorem 1.0.2 (Melchior). *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{R}}$ be an arrangement of $d \geq 3$ lines with $t_d = 0$, then*

$$t_2 \geq 3 + \sum_{r \geq 3} (r-3)t_r.$$

Melchior's result provides a small improvement of Gallai's Theorem – now we have not just one but at least three double intersection points under the assumption that our arrangement is not a pencil. After Melchior's result, it was an open problem to decide whether a similar inequality can hold for line arrangements in the complex projective plane. It turns out there is a similar inequality. In 1983 Hirzebruch [22] proved a ground-breaking result in the combinatorial theory of line arrangements, which is in fact a by-product of his work on surfaces of general type.

Theorem 1.0.3 (Hirzebruch). *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 6$ lines such that $t_d = t_{d-1} = 0$, then*

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r \geq 4} (r-4)t_r.$$

For both inequalities mentioned above it is natural to investigate examples of line arrangements for which we obtain equalities (or we are close to get equalities), and to understand geometric properties characterizing such arrangements. It turns out that these line arrangements are extremal in their nature, for instance they possess a large number of triple intersection points – as simplicial line arrangements do – or, in the case of Hirzebruch's inequality, these examples are related to finite reflection groups.

Somehow surprisingly, these extremal properties have also other manifestations in different areas of the current research fields in commutative algebra, or algebraic geometry. In order to give some feeling, let us present some recent connections. In the theory of matroids of arbitrary rank, hyperplane arrangements can be viewed as representable matroids, and we have some interesting applications, for instance certain classes of hyperplane arrangements are related to the so-called wonderful compactifications in the sense of de Concini-Procesi and the combinatorial Chow rings [1]. In topology, there is an interesting problem devoted to the complex compactifications of the complements of line arrangements (in this case we have non-trivial topology), for instance Rybnikov found two interesting complex line arrangements of 13 lines (based on MacLane combinatorics) being combinatorially equivalent, but having non-homeomorphic complements (their fundamental groups are different). This means that the global topology of line arrangements is not combinatorial in its nature [2]. At this stage the mentioned areas of research might appear quite remote from the viewpoint of the present thesis, but we want to emphasize that the subject of our interest has broad connections and applications in various branches of mathematics.

Now we would like to present a short outline and the main results of the thesis. Our hope is to provide an interesting merger of combinatorial and algebraic methods that allows to understand such beautiful geometrical objects as line arrangements are. The Leitmotif of our research is a family of Böröczky line arrangements – this is an interesting classical object strictly related to the problem of classification of line arrangements in the real projective plane possessing the maximal possible number of triple intersections points, quite along the lines of the Green and Tao beautiful results in [20].

The first part of this thesis is devoted to basics on combinatorics of line arrangements in the projective planes and their applications, mostly in the context of the freeness of hyperplane arrangements in the projective spaces, and the so-called containment problem.
In Chapter 3, we study *parameter spaces* of certain Böröczky line arrangements for 13, 14, 16, 18, and 24 lines. In Chapter 4 we look from the combinatorial point of view on the containment problem $I^{(3)} \subset I^2$, where I is the radical ideal of a finite set of points in the projective plane, and $I^{(m)}$ denotes the m-th symbolic power. In recent years the containment problem (in general) gained a lot of attention among the renowned researchers whose list include L. Ein, R. Lazarsfeld, C. Huneke, B. Harbourne, and many others. In this circle of ideas there are still many open questions to explore. In this thesis, we look at the containment problem above from the viewpoint of the combinatorics of line arrangements in the complex or real projective plane emphasizing the role of extreme point-line configurations in the sense of the orchard problem. This background reveals an important role played by Böröczky's family.
In the last chapter, we study the freeness of line arrangements. Let us recall that to a line arrangement (or in general a hyperplane arrangement) in a projective plane one can associate certain (sub)module of polynomial derivations that encodes combinatorial features of the arrangement. It turns out that if the mentioned (sub)module is a free module, then the arrangement has many interesting properties and applications in different fields (for instance the brand new field of research on unexpected hypersurfaces in projective spaces). We study the freeness of Böröczky's family of line arrangements, and variations on the Böröczky construction.

At the end of the thesis, we study extensions to the supersolvability – a special class of free line arrangements having very restrictive combinatorics. It turns out that our method allows to construct new examples of supersolvable line arrangements that might be applied in further research. Since the core of our thesis is rather technical, and it is difficult to present the main results word-by-word, let us show here an outline:

- In Chapter 3, we study the parameter spaces of certain Böröczky's arrangement of lines, namely for $n = 13, 14, 16, 18, 24$ lines. We show that these parameter spaces are in fact high genus curves. Combined with the famous Falting's result on the number of rational points on varieties of general type this implies that to construct construct combinatorics of the mentioned Böröczky's line arrangements over the rational numbers is either extremely hard or impossible. This path of studies was suggested by B. Harbourne in order to verify whether one can find new (counter)examples to the containment problem. In the context of the containment problem, we show that the radical ideals I_3 of the triple intersection points of Böröczky's line arrangements up to 11 lines satisfy the containment $I_3^{(3)} \subset I_3^2$. In chapter 4, we study a natural combinatorial problem related to the containment problem which can be described shortly as follows. Suppose that we have two line arrangements in the complex projective plane having the same numbers of the intersection points of each type. Then it is natural to wonder if certain containment relation does (or does not) hold for the radical ideal of some intersection points of the first arrangement, then the containment holds (or does not hold) for the corresponding set of singular points of the second arrangement. We show, using a result due to Bokowski and Pokora, that the question has, in general, a negative answer: the containment relations are in fact not combinatorial in their nature.

- In Chapter 5, we study the notion of free line arrangements. Our main result for this part tells us that Böröczky's line arrangements of $n \geq 7$ lines are not free in the sense of Saito. In Section 5.2, we study supersolvability numbers. Given a line arrangement $\mathcal{L}$, it can be extended to a supersolvable arrangement by adding new lines. The question is what is the minimal number of lines required to obtain a supersolvable arrangement. Let us recall that a line arrangement is supersolvable in the sense of Stanley if there exists an intersection point (called a modular point) such that if we join this point with another intersection point, then the line defined by these two points is contained in the set of lines building up the arrangement. In particular, we show that for Böröczky's arrangement of $n = 6k$ lines the mentioned number is less or equal to $6k^2 - 6k$, which shows somehow that this particular subfamily of arrangements is far away from the supersolvable world.

At the end, I would like to thank my advisors, prof. dr hab. Tomasz Szemberg and Dr. habil. Piotr Pokora, for hours of discussions and a plenty of essential comments that allowed to improve the work. I would also like to thank Barbara Kabat, Zbigniew Kabat, Grzegorz Malara, Marek Janasz, Magdalena Lampa-Baczyńska, Łucja Farnik, and Beata Gryszka for their meaningful mental support.

Chapter 2

Preliminaries

2.1 Combinatorics of line arrangements

In the thesis, we will consider projective spaces $\mathbb{P}^n_{\mathbb{K}}$ defined over an arbitrary field $\mathbb{K}$.

Definition 2.1.1. A set of the form $\ell = \{[x : y : z] : ax + by + cz = 0\} \subseteq \mathbb{P}^2_{\mathbb{K}}$ for some $a, b, c \in \mathbb{K}$, not all zero, will be called a line. A line arrangement $\mathcal{L} \subseteq \mathbb{P}^2_{\mathbb{K}}$ is a finite set of two or more lines.

Definition 2.1.2 (Weak combinatorics)**.** In the thesis, the weak combinatorics of a line arrangement $\mathcal{L}$ is the vectors whose entries are the number of lines in $\mathcal{L}$ and their points of incidence, namely $(d; t_2, t_3, ..., t_d)$, where for $m \geq 2$ an m-point of $\mathcal{L}$ is a point where exactly m lines from the arrangement meet, and the number of m-points is denoted by t_m.

Remark 2.1.3. In combinatorial approach towards line arrangements, the notion of *the combinatorics of a given line arrangement* means usually something different, i.e., this is not only the lines and the intersection points, but also the way how the intersection points are distributed on each line from the arrangement, and so on. Thus, it means the *intersection lattice* $L(\mathcal{L})$.

Now let us present a couple of examples.

Example 2.1.4. In the real projective plane, any line arrangement $\mathcal{L}$ provides a partition of $\mathbb{P}^2_{\mathbb{R}}$. Denote by $CM(\mathcal{L})$ the complement of $\mathcal{L}$ in $\mathbb{P}^2_{\mathbb{R}}$. As we can see, $CM(\mathcal{L})$ consists of the union of disjoint polygons. If each polygon is a triangle, then we say that $\mathcal{L}$ is *simplicial.* Simplicial line arrangements play an important role in combinatorics. However, these objects are not completely classified yet! One of the simplest known examples of a simplicial line arrangement is the complete quadrangle $\mathcal{A}_1(6)$ defined by the zeros of $xyz(x-y)(x-z)(y-z)$. It is an arrangement of 6 lines with $t_2 = 3$ and $t_3 = 4$. Figure 2.1 shows an affine model of the arrangement.

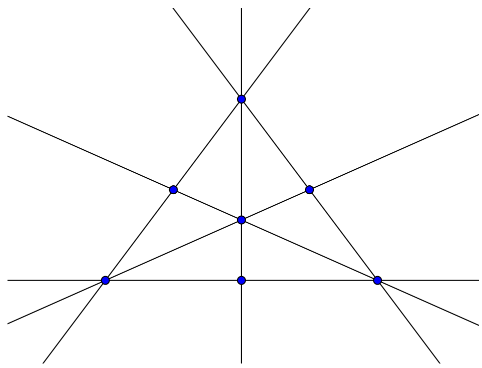

Figure 2.1: $\mathcal{A}_1(6)$ simplicial arrangement.

Example 2.1.5. Consider $\mathbb{F}_{p^n}$ for some $n > 0$ and a prime number p. It is well-known that the set of $(p^n)^2 + p^n + 1$ lines in $\mathbb{P}^2_{\mathbb{F}_{p^n}}$ forms an arrangement with $t_{p^n+1} = (p^n)^2 + p^n + 1$, and other t_m's equal to zero. We call such an arrangement a finite projective plane arrangement. If now $p = 2$ and $n = 1$, we obtain the famous Fano plane consisting of 7 lines and $t_3 = 7$.

Note that the Fano plane is one of very few known arrangements with triple points only.

Counting (intersections of) pairs of lines in two different ways, we see that any arrangement $\mathcal{L}$ of d lines in the projective plane satisfies the following combinatorial equality

$$\binom{d}{2} = \sum_{r \geq 2} \binom{r}{2} t_r. \tag{2.1}$$

This is purely combinatorial – it holds over any field. However, this combinatorial equality is a rather weak tool and it might be not enough to decide whether certain combinatorics can be constructed. For example, the Fano combinatorics, i.e., $(d; t_2, t_3) = (7, 0, 7)$, can be realized over a field $\mathbb{K}$ if and only if $\operatorname{char} \mathbb{K} = 2$. Another, almost combinatorial fact, is the de Bruijn-Erdös Theorem, see [8].

Theorem 2.1.6 (de Bruijn-Erdös)**.** *Let $\mathcal{L} \subseteq \mathbb{P}^2_{\mathbb{K}}$ be any arrangement of d lines such that $t_d = 0$. Then*

$$\sum_{r \geq 2} t_r \geq d,$$

and the equality holds if and only if $\mathcal{L}$ is either a Hirzebruch's quasi-pencil with $t_{d-1} = 1$ and $t_2 = d - 1$ or a finite projective plane arrangement.

2.1.1 Böröczky's arrangements of lines

In this subsection, we describe the main construction, namely Böröczky's arrangements $\mathcal{B}_n$ which were introduced in [19, Example 2]. Following this example, we present here an outline of the construction.

Consider a regular $2n$-gon inscribed in the unit circle in the real affine plane. Let us fix one of the $2n$ vertices and denote it by Q_0. By Q_α we denote the point arising by the rotation of Q_0 around the center of the circle by angle α.

Then we take the following set of lines

$$\mathcal{B}_n = \left\{ Q_\alpha Q_{\pi-2\alpha}, \text{where } \alpha = \frac{2k\pi}{n} \text{ for } k = 0, \ldots, n-1 \right\}.$$

If $\alpha \equiv (\pi - 2\alpha)(\text{mod } 2\pi)$, then the line $Q_\alpha Q_{\pi-2\alpha}$ is the tangent to the circle at the point Q_α. The arrangement $\mathcal{B}_n$ has $\lfloor \frac{n(n-3)}{6} \rfloor + 1$ triple points by [19, Property 4]. We denote the set of these triple points by $\mathbb{T}_n$.

Example 2.1.7 (Böröczky arrangement of 13 lines)**.** In Figure 2.2 we present the Böröczky arrangement for $n = 13$ with distinguished point Q_0. In this case we have exactly 22 triple intersection points and these points form a very interesting arrangement: one of the lines contains 6 of these triple points, and each of the remaining 12 lines contains exactly 5 triple points.

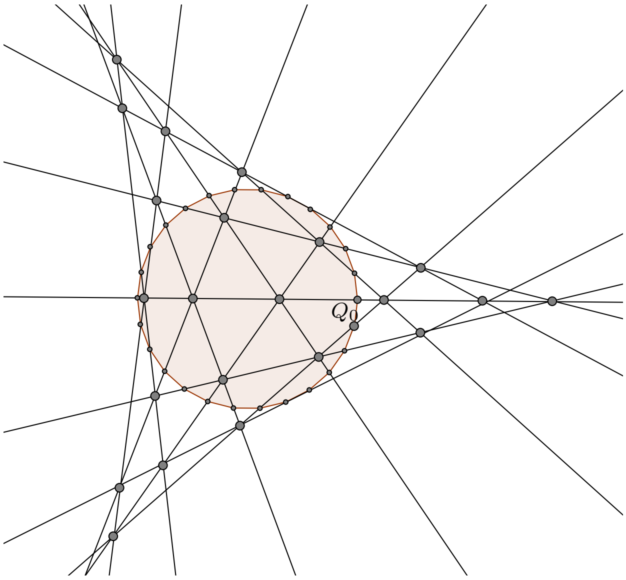

Figure 2.2: The regular 26-gon and the 13 lines of Böröczky.

In the sequel we will need the following simple fact concerning the distribution of triple points on the arrangement lines.

Proposition 2.1.8. *Every line in the $\mathcal{B}_n$ arrangement contains at least $\lfloor \frac{n-3}{2} \rfloor$ triple points and there exists a line containing at least one more triple point.*

Proof. By construction the triple points are distributed on the arrangement lines almost uniformly, that means that the difference between the number of points from $\mathbb{T}_n$ on two arrangement lines is at most 1. Let s be the minimal number of triple points on an arrangement line. Then it must be

$$\frac{sn}{3} \leq 1 + \lfloor \frac{n(n-3)}{6} \rfloor \text{ and } \lfloor \frac{n(n-3)}{6} \rfloor \leq \frac{(s+1)n}{3}$$

and the claim follows. □

We derive the following, very useful, consequence of Proposition 2.1.8.

Corollary 2.1.9. *For a fixed $n \geq 8$ let C be a plane curve (possibly reducible and non-reduced) of degree d passing through every point in the set $\mathbb{T}_n$ with multiplicity at least 3. Then $d \geq n$. Moreover, if $d = n$, then C is the union of all arrangement lines in $\mathcal{B}_n$.*

Proof. Assume to the contrary, that $d < n$. By Proposition 2.1.8 an arrangement line ℓ contains at least $\lfloor \frac{n-3}{2} \rfloor$ triple points. If ℓ is not a component of C, then it must be, by Bézout Theorem,

$$n > d \geq 3\lfloor \frac{n-3}{2} \rfloor.$$

It follows that

$$n + 3 > 3\lfloor\frac{n}{2}\rfloor,$$

which contradicts the assumption $n \geq 8$. □

2.2 Freeness of hyperplane arrangements and of divisors

In this section, we consider all objects over the complex numbers, even if some definitions are true over an arbitrary field. Our major reference for this section is Dimca's book [11].

Let $S = \bigoplus_k S_k = \mathbb{C}[x_0, \ldots, x_n]$ be the graded polynomial ring in $n+1$ indeterminates with complex coefficients, where S_k denotes the vector space of degree k homogeneous polynomials. Let $f \in S_d$ be a degree d polynomial and denote by J_f the corresponding Jacobian ideal generated by the partial derivatives $f_j = \frac{\partial f}{\partial x_j}$. Now we define the graded Milnor algebra $M(f) = \bigoplus_k M(f)_k = S/J_f$. The graded module of all Jacobian syzygies (algebra of relations) is defined by

$$AR(f) = \{r = (a_0, a_1, ..., a_n) \in S^{n+1} : a_0 f_0 + a_1 f_1 + ... + a_n f_n = 0\}.$$

To each Jacobian relation $r \in AR(f)$, one can associate a derivation

$$\delta(r) = a_0 \partial_{x_0} + a_1 \partial_{x_1} + ... + a_n \partial_{x_n}$$

of the polynomial ring S. Note that $\delta(r)$ kills f, that is $\delta(r)(f) = 0$. The set of all derivations that kill f is denoted by $D_0(f)$, a graded S-module isomorphic to the module $AR(f)$. One can consider the Euler derivation

$$\delta_E = x_0 \partial_{x_0} + x_1 \partial_{x_1} + ... + x_n \partial_{x_n}$$

and then the graded S-module

$$D(f) = S \cdot \delta_E \oplus D_0(f)$$

consists of all derivations δ of the polynomial ring S preserving the principle ideal (f).

Now we are ready to present one of possible freeness characterizations.

Theorem 2.2.1 (Freeness of divisors). *Let V be a divisor in $\mathbb{P}^n_{\mathbb{C}}$ defined by a homogeneous polynomial f. We say that V is free (f is free) if one of the following equivalent conditions is satisfied:*

- *the module $AR(f)$ is a free graded S-module;*

- *the module $D_0(f)$ is a free graded S-module;*

- *the module $D(f)$ is a free graded S-module.*

Let f be such that $AR(f)$ is free. Then the rank of S-module $AR(f)$ is n. Let $r_i = (r_{i0}, ..., r_{in}) \in AR(f) \subset S^{n+1}$ for $i = 1, ..., n$ be a homogeneous basis of $AR(F)$ with deg $r_i = d_i$. We call the integers d_i the exponents of f. Consider also the vector $r_0 = (r_{00}, ..., r_{0n}) = (x_0, ..., x_n)$, which is not in the module of $AR(f)$, but it corresponds to the Euler derivation. In order to show the freeness of certain divisor, one can use the following Saito's result – this can be considered as a folklore result, but we took it from Dimca's book [11].

Theorem 2.2.2 (Saito)**.** *The homogeneous Jacobian syzygies $r_i \in AR(f)$ for $i = 1, ..., n$ form a basis of this S-module if and only if $\varphi(f) = cf$, where $\varphi(f)$ is the determinant of $(n+1)\times(n+1)$ matrix $\Phi(f) = [r_{ij}]_{i,j=0,...,n}$ and c is a non-zero constant.*

Now we consider a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^{n+1}$ which is the affine cone over the corresponding projective arrangement in $\mathbb{P}^n_{\mathbb{C}}$. We define the intersection lattice of $\mathcal{A}$ (in particular, such arrangements are central arrangements, i.e., $\bigcap_{H\in\mathcal{A}} H \ni \{0\}$).

Definition 2.2.3.

- A non-empty intersection X of a family of hyperplanes in $\mathcal{A}$ is called a flat of $\mathcal{A}$. Note that $\mathbb{C}^{n+1}$ itself is always a flat, the intersection of the empty family of hyperplanes.

- The intersection poset of $\mathcal{A}$ is the set $L(\mathcal{A})$ of all the flats X of $\mathcal{A}$ with the order $\leq$ defined by $X \leq Y$ if and only if $Y \subseteq X$.

- The rank function $r : L(\mathcal{A}) \to \mathbb{Z}$ is defined by $r(X) = n + 1 - \dim X = \text{codim}X$.

We write $X < Y$ if and only if $X \leq Y$ and $Y \subsetneq X$. Moreover by $X \not\lesssim Y$ we understand elements which are not comparable with respect to $\leq$, i.e., neither $X \not\lesssim Y$ nor $Y \not\lesssim X$ holds.

Definition 2.2.4. The Möbius function μ of an arbitrary poset L is defined as the unique function $\mu : L \times L \to \mathbb{Z}$ such that

- $\mu(x, x) = 1$ for any $x \in L$;

- $\sum_{x\leq z\leq y} \mu(x, z) = 0$ for all $x, y \in L$ with $x < y$;

- $\mu(x,y) = 0$ for $x \not\leq y$.

If L has a minimal element $\tilde{0}$, we set $\mu(x) = \mu(\tilde{0}, x)$. In our setting, when $L = \mathcal{L}(\mathcal{A})$, $\mathbb{C}^{n+1}$ is the unique minimal element and we have $\mu(\mathbb{C}^{n+1}) = 1$. If $H \in \mathcal{A}$, then $\mu(H) = -1$.

Definition 2.2.5. The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

and the Poincaré polynomial of a hyperplane arrangement is defined by

$$\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{r(X)}.$$

An intuitive example of a free arrangement (or divisor) is given by the class of supersolvable lattices.

Definition 2.2.6. Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{C}^{n+1}$.

- A flat $X \in L(\mathcal{A})$ is modular if $X + Y \in L(\mathcal{A})$ for any other flat $Y \in L(\mathcal{A})$, where $X + Y$ denotes the linear subspace generated by $X \cup Y$.
- The arrangement $\mathcal{A}$ is supersolvable if the intersection lattice $L(\mathcal{A})$ has a maximal chain

$$\mathbb{C}^{n+1} = X_0 < X_1 < ... < X_r = \{0\}$$

of modular flats with $r = \mathrm{rank}(\mathcal{A})$ and $C(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$ is the centre of $\mathcal{A}$.

Let us explain this notion using a down-to-earth example.

Example 2.2.7. Consider a near-pencil of d lines in the projective plane which has $t_{d-1} = 1$ and $t_2 = d-1$. If we take the vertex corresponding to multiplicity $d-1$, then we can join it with each double intersection point using a line from the arrangement – this description explains what we understand by being supersolvable in the case of projective planes. In fact, all the double points are also modular.

We have mentioned that supersolvable arrangements are free, and this follows from Jambu-Terao's result [24].

Now we focus on the case of the complex projective plane. For a line arrangement $\mathcal{A}$ its Levi graph is defined as a bipartite graph with one vertex per (singular) point and one vertex per line with an edge for every incidence between a (singular) point and a line.

Definition 2.2.8. We say that two line arrangements $\mathcal{A}$ and $\mathcal{B}$ are (strongly) combinatorially equivalent if the associated Levi graphs are isomorphic (equivalently, if the intersection posets are isomorphic).

A central problem in the theory of hyperplane arrangements is the following conjecture due to Terao.

Conjecture 2.2.9 (Terao)**.** *Let $\mathcal{A}_1$ and A_2 be two line arrangements in the complex projective plane such that the associated Levi graphs are isomorphic. Assume that $\mathcal{A}_1$ is free, then $\mathcal{A}_2$ is also free.*

We know by [13] that Terao's conjecture is true for up to 13 lines, but except that case the conjecture is widely open. We are not going to approach this conjecture since this is a very difficult problem. Let us emphasize that it is also a very elegant problem since the freeness of certain modules of derivations is expected to be encoded in combinatorics of the associated line arrangements – somehow surprisingly. As it was mentioned in Introduction, the topological counterpart of the story that combinatorics should be reflected in algebro-topological properties of certain associated objects is false, so it is extremely interesting to decide whether Terao's conjecture is true or not.

Finally, let us recall the main tool in our thesis that allows to decide whether certain line arrangement is free. This result works in the whole generality.

Definition 2.2.10. We say that $C : \{f = 0\}$ of degree $\deg f = d$ is a *free curve* with the exponents (d_1, d_2) if the minimal resolution of the Milnor algebra $M(f)$ is of the form

$$0 \longrightarrow S(-d_1 - (d-1)) \oplus S(-d_2 - (d-1)) \longrightarrow S^3(-d+1) \to S \to M(f) \to 0,$$

with $d_1 + d_2 = d - 1$.

Example 2.2.11. Consider the line arrangement given by $Q(x, y, z) = xy(x + y + z)$. The Jacobian ideal corresponding to Q is given by

$$Jac(Q) = \langle 2xy + y^2 + yz, x^2 + 2yx + xz, xy \rangle.$$

Then the minimal set of generators of $Jac(Q)$ looks as follows

$$Jac(Q) = \langle y^2 + yz, x^2 + xz, xy \rangle.$$

We compute the resolution of $Jac(Q)$. The first map $S^3(-2) \to S$ is given by the partial derivatives $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}$. Next, we are going to look at non-trivial relations among the partials. It is easy to observe that

$$x \cdot (y^2 + yz) - (y+z) \cdot xy = 0,$$

$$(x+y) \cdot xy - y \cdot (x^2 + xz) = 0,$$

so the Hilbert-Burch matrix of the first syzygies looks as follows

$$A = \begin{pmatrix} x & 0 & -y-z \\ 0 & -y & x+z \end{pmatrix}^T.$$

Obviously, we have $1 \cdot x + (-1) \cdot (-y-z) + 1 \cdot (-y) + (-1) \cdot (x+z) = 0$, so the minimal free resolution has the following form:

$$0 \to S(-3) \oplus S(-3) \xrightarrow{A} S(-2)^3 \xrightarrow{(\partial_x, \partial_y, \partial_z)} S \to M(Q) \to 0,$$

so the arrangement defined by Q is free with exponents $d_1 = d_2 = 1$.

2.3 Symbolic powers of ideals and the containment problem

In this section we assume that $\mathbb{K}$ is an arbitrary algebraically closed field.

Definition 2.3.1. Let $I \subseteq S$ be a homogeneous ideal and let $m \geq 1$ be a positive integer. The m-th symbolic power of I is defined as

$$I^{(m)} = S \cap \bigcap_{Q \in \mathrm{Ass}(I)} (I^m)_Q,$$

where $\mathrm{Ass}(\cdot)$ denotes the set of associated primes, and the intersection is taken in the field of fractions of S.

Symbolic powers are algebraic objects, but we can see their association with geometry due to glorious **Zariski-Nagata theorem**, which says, that if I is a radical ideal, then

$$I^{(m)} = \left\{ f \in I : \frac{\partial^{|\alpha|} f}{\partial x^{\alpha}} = 0 \text{ on the zeros of } I, \text{ for all } |\alpha| < m \right\}.$$

Directly from definitions of symbolic and ordinary powers of an ideal we have the following containments:

$$I = I^{(1)} \supseteq I^{(2)} \supseteq I^{(3)} \supseteq \dots$$

and

$$I = I^1 \supseteq I^2 \supseteq I^3 \supseteq \dots$$

It is natural to ask if there are some containment relations between the members of symbolic family of powers of ideals and the members of family of their ordinary powers. In one direction this relation is quite elementary.

Proposition 2.3.2. *Let $I \subset \mathbb{K}[x_0, \dots, x_n]$ be a radical homogeneous ideal. Then the containment*

$$I^r \subset I^{(m)}$$

holds if and only if $r \geq m$.

The reverse containment is much more subtle. It has been established in 2001 by Ein, Lazarsfeld and Smith [16] under the additional assumption that $\operatorname{char} \mathbb{K} = 0$. This assumption is not essential as proved by Hochster and Huneke [23]. Since we are interested in the first line in ideals of points, we present the Containment Theorem in a somewhat simplified version.

Theorem 2.3.3 (Containment Theorem)**.** *Let $I \subset \mathbb{K}[x_0, \dots, x_n]$ be a homogeneous ideal. Then the containment*

$$I^{(m)} \subset I^r$$

holds for all $m \geq nr$.

The key feature of Theorem 2.3.3 is that the statement does not depend on I!

It is natural to wonder to what extent the lower bound nr is optimal. Clearly, it cannot be optimal for any I. For some classes of ideals it is even very far from being optimal. For example, if I is a complete intersection ideal, then its symbolic and ordinary powers agree and one does not need the factor n. The same happens for other classes of ideals.

Example 2.3.4 (Edge ideals of bipartite graphs)**.** Let G be a simple graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. The *edge ideal* $I(G)$ of G is defined by

$$I(G) = (x_i x_j : \ x_i x_j \in E(G)) \subset \mathbb{K}[x_1, \dots, x_n].$$

Simis, Vasconcelos and Villarreal showed in [35] that all symbolic and ordinary powers of $I(G)$ coincide if and only if G is bipartite.

Huneke asked around 2006 if the containment $I^{(3)} \subset I^2$ holds for radical ideals of points in the projective plane. Note that the containment $I^{(4)} \subset I^2$ follows from Theorem 2.3.3. This question has motivated a considerable part of this thesis. Before, it has motivated a lot of research, and led Bocci, Harbourne and Huneke to formulate the following, conjectural, improvement to the lower bound in Theorem 2.3.3.

Conjecture 2.3.5. *Let I be saturated ideal of a finite set of reduced points in $\mathbb{P}^n$. Then the containment*

$$I^{(m)} \subset I^r$$

holds for $m \geq nr - (n-1)$.

Strangely enough, it turns out that Conjecture 2.3.5 fails in its original setting due to Huneke, i.e., it fails for ideals of points in $\mathbb{P}^2$. The first non-containment result was exhibited by Dumnicki, Szemberg, and Tutaj-Gasińska in [15]. They study set of 12 intersection points of 9 lines arranged in the so called dual Hesse arrangement. This arrangement cannot be realized over the reals and it is rigid, i.e., any arrangement of 9 lines intersecting by 3 in 12 points, is projectively equivalent to the dual Hesse arrangement, see [28] for a direct argument.

The first real non-containment example was provided in [7]. The construction there is based on the Böröczky arrangement of 12 lines which is defined over the reals but not over the rational numbers. However, it turned out that in this example the 12 lines can be slightly modified so that they can be defined over the rational numbers. This path of investigations was followed by Lampa-Baczyńska and Szpond in [27]. They introduced the notion of a parameter space for a line arrangement. We will follow closely their ideas in the study of other arrangements in the Böröczky series of examples in the next chapter.

Chapter 3

Böröczky arrangements

3.1 Parameter spaces of some Böröczky line arrangements

In order to put our research in the perspective, we begin by recalling what is usually understood by a moduli space of line arrangements and explaining how our parameter spaces fit into the picture. In this set-up it is convenient and customary to consider a line ℓ in the projective plane as a point in the dual projective plane $(\mathbb{P}^2_{\mathbb{C}})^*$.

We say that two line arrangements $\mathcal{A}$ and $\mathcal{A}'$ are *combinatorially equivalent* $\mathcal{A} \sim \mathcal{A}'$ if their intersection lattices $L(\mathcal{A})$ and $L(\mathcal{A}')$ are isomorphic. That means that there exists a bijection $\varphi : \mathcal{A} \to \mathcal{A}'$ (extending naturally to intersection lattices) that preserves the lattice order, i.e.,

$$B \leq C \text{ if and only if } \varphi(B) \leq \varphi(C)$$

for all $B, C \in L(\mathcal{A})$.

If $\mathcal{A}$ is an ordered arrangement, i.e., the lines in $\mathcal{A}$ are numbered, then two ordered arrangements are *ordered combinatorially equivalent* if the bijection φ respects the order.

The following definition is taken from [2].

Definition 3.1.1 (Moduli spaces of ordered line arrangements)**.** The moduli space of arrangements of ordered n lines with the fixed intersection poset $L(\mathcal{A})$ is defined as

$$\mathcal{M}_{\mathcal{A}} = \left\{ \mathcal{B} \in (\mathbb{P}^{2\,*}_{\mathbb{C}})^n : \mathcal{B} \sim \mathcal{A} \right\} / \operatorname{PGL}(3, \mathbb{C}),$$

where the action of $\mathrm{PGL}(3,\mathbb{C})$ is defined naturally as

$$\mathrm{PGL}(3,\mathbb{C}) \times (\mathbb{P}^{2\,*}_{\mathbb{C}})^n \ni (g, (\ell_1, ..., \ell_n)) = (g\ell_1, ..., g\ell_n) \in (\mathbb{P}^{2\,*}_{\mathbb{C}})^n.$$

We emphasize the adjective *ordered* in Definition 3.1.1 since we fix the order of lines – in general one allows to have unordered lines and then we must take an additional quotient by an appropriate symmetric group. The following Example shows that it is important to distinguish between ordered and unordered arrangements.

Example 3.1.2. Let $\omega_{\pm} = \frac{1\pm\sqrt{-3}}{2}$. We consider two ordered sets of 8 lines $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$, where $\mathcal{A}^{\pm} = \{\ell_1, ..., \ell_5, \ell_6^{\pm}, \ell_7^{\pm}, \ell_8^{\pm}\}$ given by

$$\ell_1 : x = 0, \quad \ell_2 : y = 0, \quad \ell_3 : z = 0, \quad \ell_4 : y - x = 0,$$

$$\ell_5 : z - x = 0, \quad \ell_6^{\pm} : z + \omega_{\pm} y = 0, \quad \ell_7^{\pm} : z + \omega_{\pm}^2 x + \omega_{\pm} y = 0, \quad \ell_8^{\pm} : z - x - \omega_{\pm}^2 y = 0.$$

It is known, see for instance [31], that the moduli space (after fixing $L(\mathcal{A})$) is

$$\mathcal{M}_{\mathcal{A}} = \{\mathcal{A}^{+}, \mathcal{A}^{-}\},$$

so it consists of exactly two points.

However, passing to unordered arrangements, the moduli space has just one element, see [2, Example 1.7].

The combinatorics from Example 3.1.2 plays an important role in the topology of complements and the so-called Zariski pairs, but we are not going to discuss the details here.

Parameter spaces of line arrangements considered here are a much less formal object. The idea behind is very simple. Knowing the intersection poset of the arrangement, one reconstructs the arrangement starting from the scratch. Here, we start always with four points in general position. We may, and do, choose their coordinates in a convenient way. Then we construct additional lines and their intersection points, following the receipt encoded in the intersection poset. Whenever there is some ambiguity in making the next step in the construction, we introduce a new parameter, and continue the construction. Typically, at the end, we arrive at a number of constraints (equations) involving the parameters, which must be satisfied so that we obtain the whole intersection poset. In this approach some values of parameters correspond to degenerate arrangements, where some lines or points fall together. Parameter spaces can be thus considered as some particular compactifications of *realization spaces* (see [2]), which typically are quasi-projective varieties. It seems that the approach presented here has been pioneered by Lampa-Baczyńska and Szpond in [27] and we thank them for teaching us their techniques.

Later on, in our considerations we will be interested in finding rational points of the parameter spaces of line arrangements. In the case of Böröczky families of line arrangements, the parameter spaces turn out to be curves of genus $g \geq 2$. The celebrated Mordell Conjecture (solved by Faltings) tells us that curves of genus $g \geq 2$ defined over the rationals have only finitely many rational points. However, Faltings's proof does not tell us how we can find these rational point in an effective way. It turns out that old methods developed into the direction of the proof of Mordell Conjecture are very useful for searching the rational points. Here we are going to use Chabauty's method – a very nice description of this method is presented by Poonen in [32]. The idea standing behind Chabauty's method is to use p-adic extensions, but we do not want to go into details since it involves some technicalities not related to the core of the thesis. However, we are going to use MAGMA's command which allows to apply this method and find rational points in our cases.

Our key motivation for this search of rational points on some algebraic curves stems from the desire to construct new rational non-containment examples for the $I^{(3)} \subset I^2$ problem. Such rational non-containment examples seem to be quite rare. We pass to this question in Chapter 4.

By $\mathbb{B}_n$ we denote arrangements which preserve all incidences of the original construction of Böröczky arrangement $\mathcal{B}_n$ on n lines. Thus the arrangements $\mathbb{B}_n$ are in particular less symmetric then the original arrangements $\mathcal{B}_n$.

3.1.1 Construction of $\mathbb{B}_{13}$

The original Böröczky construction of $\mathcal{B}_{13}$ uses trigonometric functions and is based on vertices of a regular 26-gon. The core of the construction of $\mathbb{B}_{13}$ is the set of four general points in the projective plane and five lines joining certain pairs of these points. To simplify the calculations we begin our construction with the four fundamental points:

$$P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1), \quad P_4 = (1 : 1 : 1).$$

Then we take the following lines

$$P_1P_4: \ z-y=0, \ \ P_1P_2: \ z=0, \ \ P_1P_3: \ y=0, \ \ P_2P_4: \ z-x=0, \ \ P_3P_4: \ y-x=0.$$

It gives us the coordinates of intersection points

$$P_5 = P_3P_4 \cap P_1P_2 = (1:1:0) \ \text{ and } \ P_6 = P_1P_3 \cap P_2P_4 = (1:0:1).$$

From now on we need to introduce a parameter. We choose a point $P_7 \in P_1P_4$, distinct from all previous points. We write its coordinates with the parameter $a \neq 1$ and $a \neq 0$

$$P_7 = (a:1:1).$$

The construction up to this point is depicted in Figure 3.1.

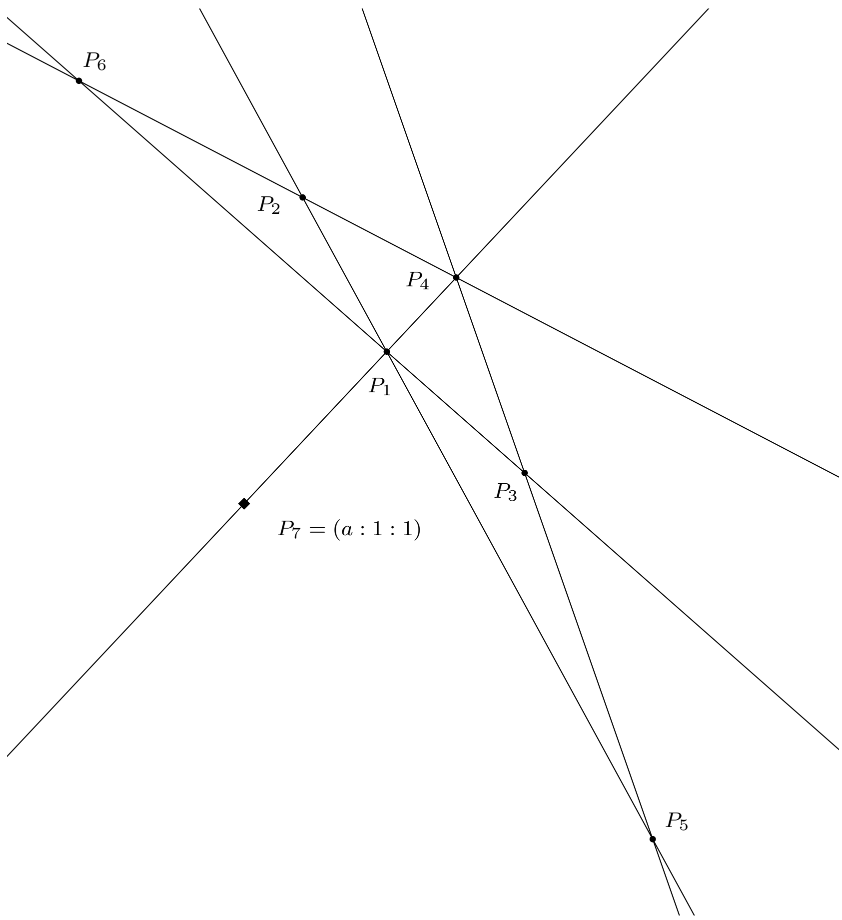

Figure 3.1: Construction of $\mathbb{B}_{13}$ at the stage of choosing parameter a

Continuing the construction, we obtain the following equations of lines

$$P_3P_7: \ ay-x=0, \qquad P_2P_7: \ x-az=0.$$

In the next step, we take the following points:

$$P_8 = P_3P_7 \cap P_1P_2 = (a : 1 : 0),$$

$$P_9 = P_2P_7 \cap P_1P_3 = (a : 0 : 1),$$

$$P_{10} = P_3P_4 \cap P_2P_7 = (a : a : 1),$$

$$P_{11} = P_3P_7 \cap P_2P_4 = (a : 1 : a).$$

Now we need to introduce an additional point $P_{12} \in P_1P_4$ distinct from all points from P_1 to P_{11} and depending on a new parameter $b \notin \{1, a\}$:

$$P_{12} = (b : 1 : 1).$$

Figure 3.2 indicates the construction at the current stage.

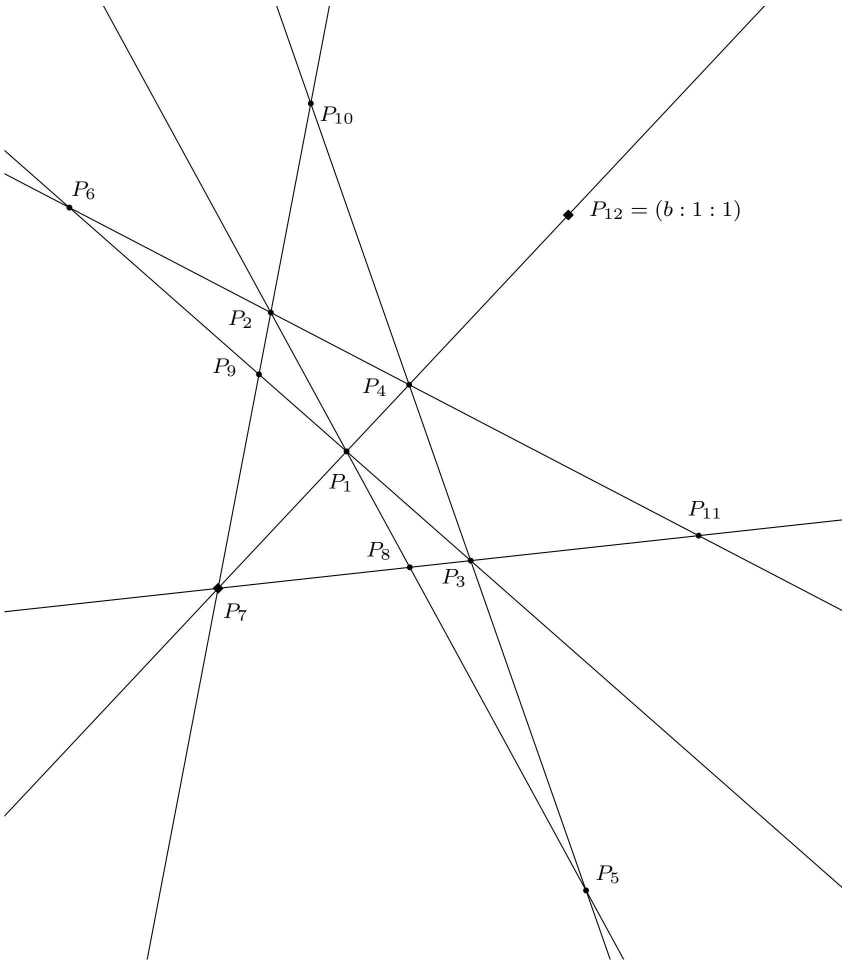

Figure 3.2: Construction of $\mathbb{B}_{13}$ at the stage of choosing parameter b

The next two lines in our construction are

$$P_{12}P_8:\ -x+ay+(b-a)z=0 \text{ and } P_{12}P_9:\ x+(a-b)y-az=0.$$

This gives us the following points

$$P_{13}=P_{12}P_8\cap P_2P_4=(a:a-b+1:a),$$

$$P_{14}=P_{12}P_9\cap P_3P_4=(a:a:a-b+1),$$

$$P_{15}=P_1P_2\cap P_{12}P_9=(b-a:1:0),$$

$$P_{16}=P_{12}P_8\cap P_1P_3=(b-a:0:1).$$

The last four lines of the construction are

$$P_{10}P_{15}:\ x-(b-a)y+(ab-a^2-a)z=0,$$

$$P_{16}P_{11}:\ x+(ab-a^2-a)y-(b-a)z=0,$$

$$P_{13}P_5:\ -ax+ay+(b-1)z=0,$$

$$P_6P_{14}:\ -ax+(b-1)y+az=0.$$

Finally, we obtain the remaining triple points

$$P_{17}=P_1P_4\cap P_{10}P_{15}\cap P_{11}P=(a^2+b-ab:1:1),$$

$$P_{18}=P_{12}P_8\cap P_{10}P_{15}\cap P_6P_{14}=(b^2-a^2-b+a-ab:ab-a^2-a:-a^2+b-1),$$

$$P_{19}=P_2P_7\cap P_6P_{14}\cap P_{11}P=(a-ab:a-a^2:1-b),$$

$$P_{20}=P_{12}P_9\cap P_{13}P_5\cap P_{11}P=(b^2-a^2-b+a-ab:-a^2+b-1:ab-a^2-a),$$

$$P_{21}=P_{13}P_5\cap P_{10}P_{15}\cap P_3P_7=(a-ab:1-b:a-a^2),$$

$$P_{22}=P_1P_4\cap P_{13}P_5\cap P_6P_{14}=(a+b-1:a:a).$$

It is easy to check (computing a suitable determinant) that points P_{17} and P_{22} are always the common points of given three lines, independently of the values of a and b. The situation is more complicated for points P_{18}, P_{19}, P_{20} and P_{21}. These points are triple only if parameters a and b satisfy the additional condition:

$$C_{13}(a,b): a^4-a^3b+a^2b-a^2+b^2-2ab+2a-b=0.$$

This condition is necessary for the construction to terminate successfully, in the sense that we obtain exactly 22 points in $\mathbb{T}_{13}$, distributed as follows: six points on one of the lines and of five points on each of the remaining 12 lines, as in the original Böröczky arrangement $\mathcal{B}_{13}$, see Figure 3.3.

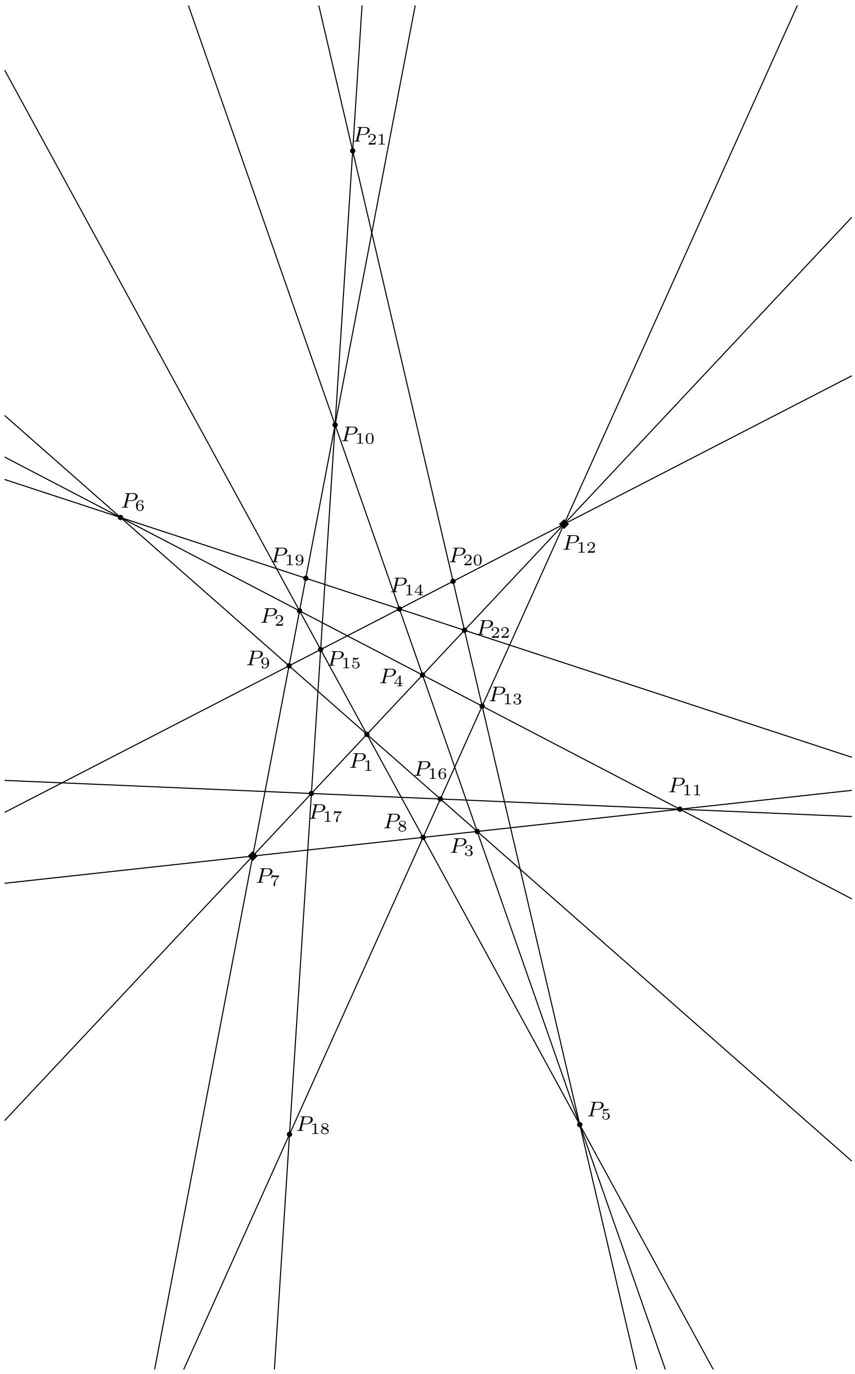

Figure 3.3: Complete construction of $\mathbb{B}_{13}$.

3.1.2 Degenerate cases of $\mathbb{B}_{13}$

In this section we check under which conditions the arrangement $\mathbb{B}_{13}$ is non-degenerate in the sense that all points and lines appearing in the construction are mutually distinct.

Proposition 3.1.3. *For $(a,b) \in \mathbb{R}^2$ such that $a \neq 0$, $a \neq 1$, $a \neq b$, and $b \neq 1$, and satisfying*

$$C_{13}(a,b) : a^4 - a^3b + a^2b - a^2 + b^2 - 2ab + 2a - b = 0 \tag{3.1}$$

the arrangement constructed in Subsection 3.1.1 consists of 13 *mutually distinct lines which intersect in exactly* 22 *triple points and* 12 *double points.*

Proof. We have already seen that the conditions $a \neq 0$, $a \neq 1$, $a \neq b$, and $b \neq 1$ are necessary. We need to check when other overlaps of points and/or lines might occur. We have the following possibilities:

i) if $a - b + 1 = 0$ then $P_4 = P_{17}$, $P_5 = P_{14} = P_{15} = P_{20} = P_{10}$ and $P_6 = P_{13} = P_{18} = P_{16} = P_{11}$,

ii) if $2a - b = 0$ then $P_8 = 0$, $P_9 = P_{16}$, $P_{11} = P_{20} = P_{13}$ and $P_{12} = P_{18} = P_{14}$,

iii) if $a^2 - ab + 2a - b = 0$ then $P_{10}P_{15} = P_{16}P_{11}$ (thus $P_{10}, P_{15}, P_{16}, P_{11}$ are collinear points),

iv) if $a^2 - b + 1 = 0$ then $P_8 = P_{18}$ and $P_9 = P_{20}$,

v) if $a^2 - a - b + 1 = 0$ then $P_7 = P_{14} = P_{21} = P_{22} = P_{19}$,

vi) if $a^3 - a^2b + ab - a - b + 1 = 0$ then $P_{17} = P_{22}$,

vii) if $a^3 - 2ab + 2a + b^2 - a^2 - b = 0$ then $P_{21} = P_{20}$ and $P_{18} = P_{19}$.

Together with main condition (3.1) it gives us

i) $-a(a-1) = 0$

ii) $-a^2(a-1)^2 = 0$

iii) $-\frac{a^3}{(a+1)^2} = 0$

iv) $-a^2(a-1)^3 = 0$

v) $-a(a-1)^4 = 0$

vi) $\frac{a^2(a-1)^3}{(a^2-a+1)^2} = 0$

vii) $a^2(a-1)(a-b) = 0$

It is easy to see, that in all these cases we get values of a and b already excluded by the assumptions. Thus, if (a, b) satisfy equation (3.1) and $a \neq 0$, $a \neq 1$, $a \neq b$ and $b \neq 1$, the construction leads to an arrangement of 13 distinct lines, which have 22 triple intersection points. □

3.1.3 Parameter space of $\mathbb{B}_{13}$ arrangements

The parameterizing curve

$$C_{13}(a,b) : a^4 - a^3b + a^2b - a^2 + b^2 - 2ab + 2a - b = 0$$

is an irreducible curve of degree 4 with one double point. This may be checked by hand or with help of a computer. Thus, the geometrical genus of C is 2. The curve may be written as

$$b^2 + b(-1 - 2a + a^2 - a^3) + a^4 - a^2 + 2a = 0.$$

Substituting b by $b - \frac{-1-2a+a^2-a^3}{2}$ and then b by $\frac{b}{2}$ we get

$$D: \ 1 - 4a + 6a^2 - 2a^3 + a^4 - 2a^5 + a^6 - b^2 = 0.$$

We denote the homogenization of D also by D. Using computer algebra programme MAGMA, we compute that the Mordell-Weil rank of the Jacobian of D has rank 0, thus Chabauty's method may be applied here. We obtain all rational points of D:

$$(1:-1:0), \quad (1:1:0), \quad (0:1:1), \quad (0:-1:1), \quad (1:1:1), \quad (1:-1:1).$$

These points correspond to the set of all rational points on C (with perhaps second coordinate changed). However, the points having $a = 1$ or $a = 0$ are excluded by the construction. Thus all possibilities lead us to the degenerated cases. Thus we have the following

Corollary 3.1.4. *The configuration $\mathbb{B}_{13}$ cannot be realized over the rational numbers.*

3.2 Construction of $\mathbb{B}_{14}$, $\mathbb{B}_{16}$, $\mathbb{B}_{18}$ and $\mathbb{B}_{24}$

For configurations $\mathbb{B}_{14}$, $\mathbb{B}_{16}$, $\mathbb{B}_{18}$ and $\mathbb{B}_{24}$ we present simplified descriptions of construction in tables. We omit some coordinates of points and some equations of lines of the configurations, because of their complicated forms. We only give the coordinates of points being the core of each construction and coordinates of points taken during the construction as parameters. We distinguish these special points using bold type font. Enough motivated reader may follow the construction step by step and find remaining coordinates and equations of lines if necessary. The idea of the construction is the same in all considered cases: we start with four fundamental points and some of the lines through them, and then we choose two points (parameters) on one of the already constructed lines. This input allows us to construct the configurations.

3.2.1 Construction of $\mathbb{B}_{14}$

The construction of $\mathbb{B}_{14}$ is based on the four fundamental points in the projective plane. We introduce here two parameters a and b such that $a \neq 1$, $b \neq 1$ and $a \neq b$. The construction goes in the following way

step 1	$P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$, $P_4 = (1:1:1)$
step 2	lines: P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4
step 3	$P_5 = P_1P_2 \cap P_3P_4$, $P_6 = P_1P_3 \cap P_2P_4$, $\mathbf{P_7} = (\mathbf{a}:\mathbf{1}:\mathbf{1}) \in \mathbf{P_1P_4}$

The construction up to now is visualised at Figure 3.4.

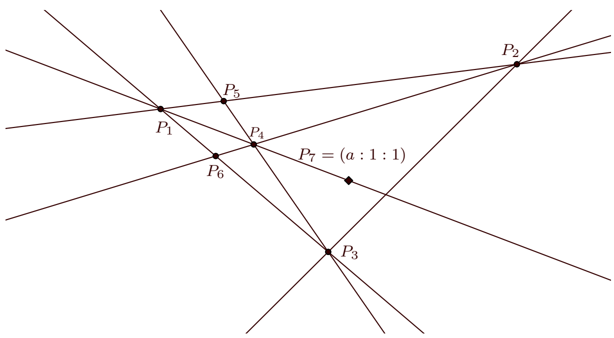

Figure 3.4: The $\mathbb{B}_{14}$ arrangement at the point of choosing the first parameter

step 4	lines: $P_5P_7,\ P_6P_7$
step 5	$P_8 = P_2P_4 \cap P_5P_7,\ P_9 = P_3P_4 \cap P_6P_7,\ P_{10} = P_1P_2 \cap P_6P_7,$ $P_{11} = P_1P_3 \cap P_5P_7,\ P_{12} = P_2P_3 \cap P_5P_7,\ P_{13} = P_2P_3 \cap P_6P_7,$ $\mathbf{P_{14} = (b : 1 : 1) \in P_1P_4}$

The construction up to now is visualised at Figure 3.5.

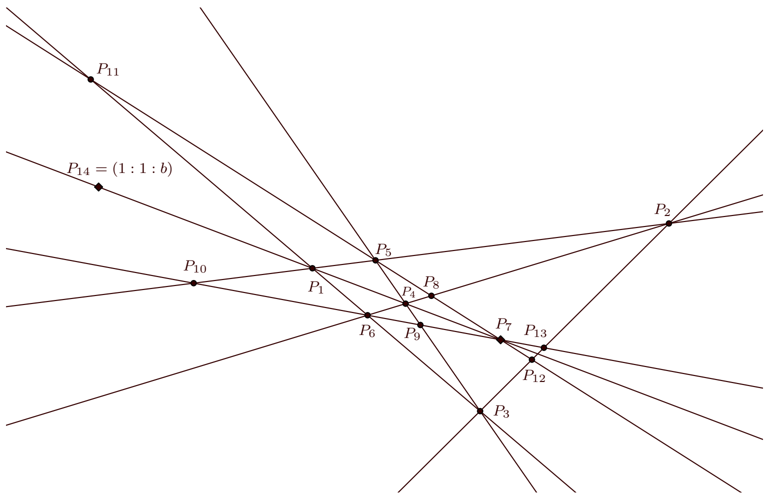

Figure 3.5: The $\mathbb{B}_{14}$ arrangement at the point of choosing the second parameter

The construction continues as follows.

step 6	lines: $P_{10}P_{14},\ P_{11}P_{14}$
step 7	$P_{15} = P_5P_7 \cap P_{10}P_{14},\ P_{16} = P_3P_4 \cap P_{10}P_{14},\ P_{17} = P_2P_4 \cap P_{10}P_{14},$ $P_{18} = P_2P_3 \cap P_{10}P_{14},\ P_{19} = P_6P_7 \cap P_{11}P_{14},\ P_{20} = P_2P_4 \cap P_{11}P_{14},$ $P_{21} = P_3P_4 \cap P_{11}P_{14},\ P_{22} = P_2P_3 \cap P_{11}P_{14}$
step 8	lines: $P_{15}P_{20},\ P_{16}P_{19},\ P_{17}P_{22},\ P_{18}P_{21}$
step 9	$P_{23} = P_1P_4 \cap P_{15}P_{20},\ P_{24} = P_1P_3 \cap P_{15}P_{20},\ P_{25} = P_1P_2 \cap P_{16}P_{19},$ $P_{26} = P_{17}P_{22} \cap P_{18}P_{21}$

This ends the construction of 14 lines with 26 triple intersection points. The complete $\mathbb{B}_{14}$ arrangement is visualized in Figure 3.6.

We sum up the discussion gathering coordinates of the points and equations of the lines in Tables 3.1 and 3.2.

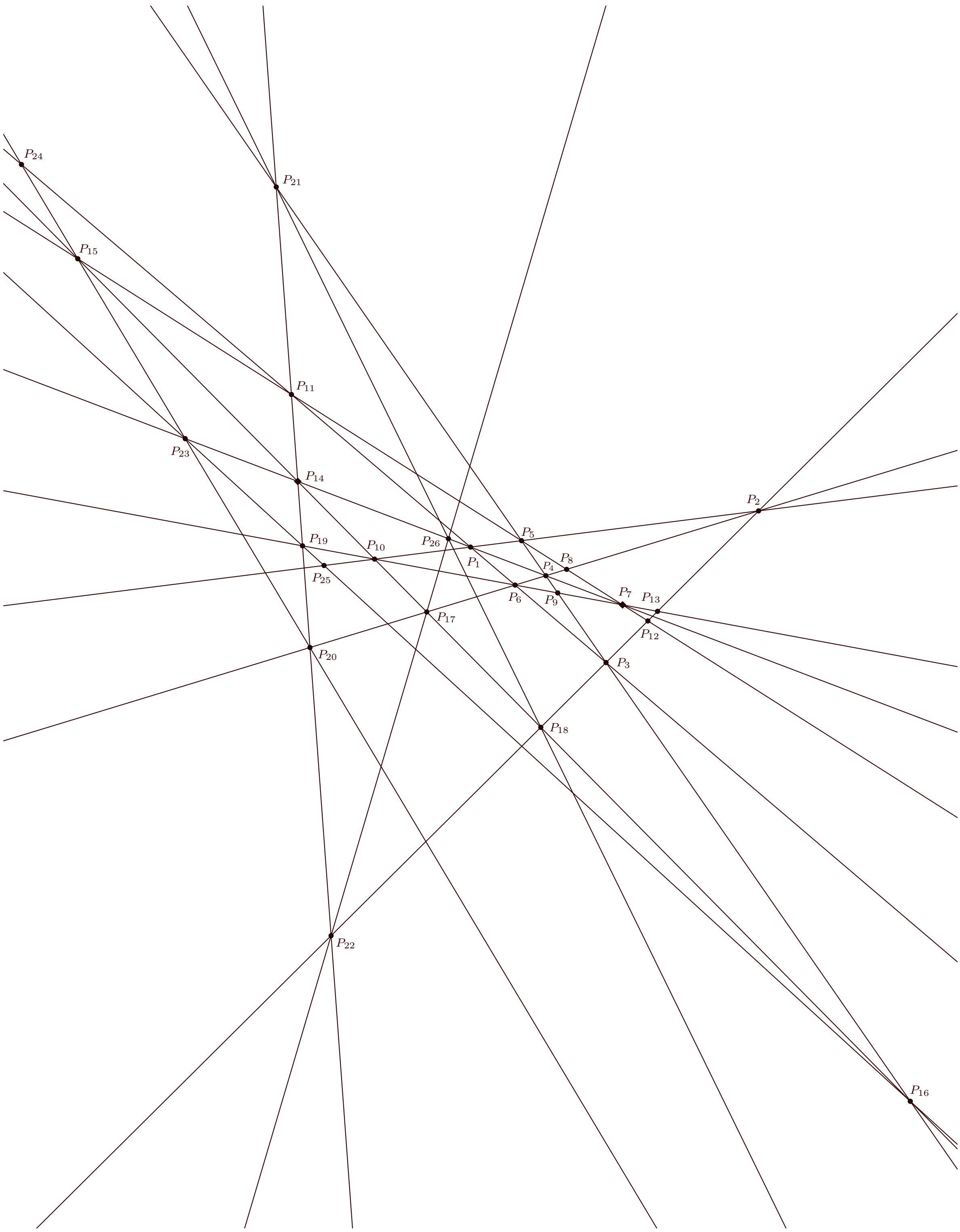

Figure 3.6: The complete $\mathbb{B}_{14}$ arrangement

Point	Coordinates
P_1	$(1:0:0)$
P_2	$(0:1:0)$
P_3	$(0:0:1)$
P_4	$(1:1:1)$
P_5	$(-1:-1:0)$
P_6	$(1:0:1)$
P_7	$(a:1:1)$
P_8	$(1:-a+2:1)$
P_9	$(1:1:-a+2)$
P_{10}	$(-a+1:-1:0)$
P_{11}	$(-a+1:0:-1)$
P_{12}	$(0:-a+1:1)$
P_{13}	$(0:-1:a-1)$
P_{14}	$(b:1:1)$
P_{15}	$(-a^2+a+b:-2a+b+2:-a+2)$
P_{16}	$(-a+b+1:-a+b+1:-a+2)$
P_{17}	$(a-1:a-b:a-1)$
P_{18}	$(0:a-b-1:a-1)$
P_{19}	$(-a^2+a+b:-a+2:-2a+b+2)$
P_{20}	$(a-b-1:a-2:a-b-1)$
P_{21}	$(-a+1:-a+1:-a+b)$
P_{22}	$(0:a-1:a-b-1)$
P_{23}	$(a^2b-2a^2+ab+2a-2b^2:-a^2+3ab-b^2-3b+2:-a^2+3ab-b^2-3b+2)$
P_{24}	$(a^3-5a^2+2ab+6a-b^2-b-2:0:-a^2+3ab-b^2-3b+2)$
P_{25}	$(-a^3+5a^2-2ab-6a+b^2+b+2:a2-3ab+b^2+3b-2:0)$
P_{26}	$(2a^3b-a^2b^2-6a^2b+2ab^2+6ab-b^2-2b:$
	$4a^3b-2a^3-4a^2b^2-9a^2b+6a^2+ab^3+7ab^2+6ab-6a-b^3-3b^2-b+2:$
	$4a^3b-2a^3-4a^2b^2-9a^2b+6a^2+ab^3+7ab^2+6ab-6a-b^3-3b^2-b+2)$

Table 3.1: Points in the $\mathbb{B}_{14}$ arrangement

P_1P_2 :	$z = 0$
P_1P_3 :	$y = 0$
P_1P_4 :	$-y + z = 0$
P_2P_3 :	$x = 0$
P_2P_4 :	$x - z = 0$
P_3P_4 :	$x - y = 0$
$P_{15}P_{20}$:	$(-a^2 + 3ab - b^2 - 3b + 2)x + (a^3 - a^2b - 3a^2 + ab + 4a + b2 - b - 2)y + (-a^3 + 5a^2 - 2ab - 6a + b^2 + b + 2)z = 0$
$P_{16}P_{19}$:	$(a^2 - 3ab + b^2 + 3b - 2)x + (a^3 - 5a^2 + 2ab + 6a - b^2 - b - 2)y + (-a^3 + a^2b + 3a^2 - ab - 4a - b^2 + b + 2)z = 0$
$P_{17}P_{22}$:	$(-2ab + a + b^2 + b - 1)x + (-a^2 + ab + 2a - b - 1)y + (a^2 - 2a + 1)z = 0$
$P_{18}P_{21}$:	$(2ab - a - b^2 - b + 1)x + (-a^2 + 2a - 1)y + (a^2 - ab - 2a + b + 1)z = 0$
$P_{10}P_{14}$:	$-x + (a - 1)y + (-a + b + 1)z = 0$
$P_{11}P_{14}$:	$x + (a - b - 1)y + (-a + 1)z = 0$
P_5P_7 :	$-x + y + (a - 1)z = 0$
P_6P_7 :	$-x + (a - 1)y + z = 0$

Table 3.2: Lines in the $\mathbb{B}_{14}$ arrangement

The point P_{23} lies additionally on the line $P_{16}P_{19}$ and the point P_{26} lies on the line P_1P_4 for any value of a and b. However, there exist points, which become triple only for certain values of parameters a and b. These points are P_8, P_9, P_{12}, P_{13}, P_{24} and P_{25}. The conditions to have them triple (i.e., to assure additional incidences) are

$$\begin{aligned} P_8 &\in P_2P_4 \cap P_5P_7 \cap P_{18}P_{21}, \\ P_9 &\in P_3P_4 \cap P_6P_7 \cap P_{17}P_{22}, \\ P_{12} &\in P_2P_3 \cap P_5P_7 \cap P_{16}P_{19}, \\ P_{13} &\in P_2P_3 \cap P_6P_7 \cap P_{15}P_{20}, \\ P_{24} &\in P_1P_3 \cap P_{18}P_{21} \cap P_{15}P_{20}, \\ P_{25} &\in P_1P_2 \cap P_{16}P_{19} \cap P_{17}P_{22}. \end{aligned}$$

We obtain the following algebraic conditions for parameters a and b:

- $2a - 3a^2 + a^3 + ab - b^2 = 0$,
- $(-2 + a)(2a - 3a^2 + a^3 + ab - b^2) = 0$,
- $(-2 + 2a - b)(-1 + b)(2a - 3a^2 + a^3 + ab - b^2) = 0$.

Thus the parameter space for configurations $\mathbb{B}_{14}$ is the curve

$$C_{14}(a, b) : 2a - 3a^2 + a^3 + ab - b^2 = 0.$$

Parameter space of configurations $\mathbb{B}_{14}$

We now take a closer look at the curve parametrizing the configurations $\mathbb{B}_{14}$, i.e.,

$$C_{14}(a,b) : 2a - 3a^2 + a^3 + ab - b^2 = 0.$$

Its smooth of geometric genus is 1. By substituting $a \mapsto a/4, b \mapsto (a+b)/4$ we get the curve in Weierstrass form

$$D : b^2 = a^3 - 11a^2 + 32a.$$

The curve D has the following rational points:

$$(0,0), \quad (4,-4), \quad (4,4), \quad (8,-8), \quad (8,8).$$

Thus the closure of C_{14} in the projective plane has the following rational points:

$$(0:0:1), \quad (1:1:1), \quad (1:0:1), \quad (2:2:1), \quad (2:0:1), \quad (0:1:0).$$

Each of them leads us to a degenerated case hence no $\mathbb{B}_{14}$ configuration can be obtained over the rational numbers.

3.2.2 Construction of $\mathbb{B}_{16}$

The core of configuration $\mathbb{B}_{16}$ are the four fundamental points. We need two parameters a and b such that $a \neq 1$, $a \neq 0$, $b \neq 0$, $b \neq 1$ and $a \neq b$. We present the construction step by step.

step 1	$P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$, $P_4 = (1:1:1)$
step 2	lines: P_1P_4, P_2P_4, P_3P_4
step 3	$\mathbf{P_5 = (a:1:1) \in P_1P_4}$

The construction up to now is visualised at Figure 3.7.

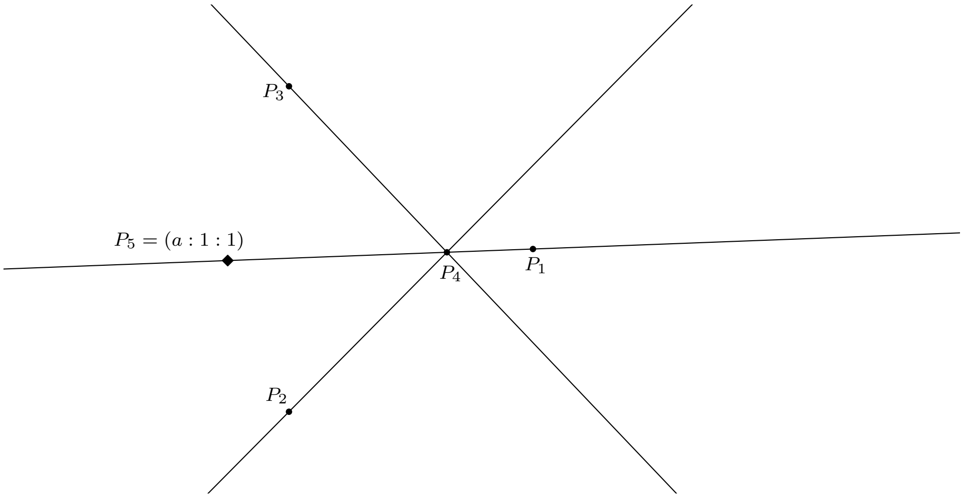

Figure 3.7: The $\mathbb{B}_{16}$ arrangement at the point of choosing the first parameter

step 4	lines: P_2P_5, P_3P_5
step 5	$P_6 = P_2P_4 \cap P_3P_5$, $P_7 = P_3P_4 \cap P_2P_5$
step 6	lines: P_1P_6, P_1P_7
step 7	$P_8 = P_3P_5 \cap P_1P_7$, $P_9 = P_2P_4 \cap P_1P_7$, $P_{10} = P_1P_6 \cap P_2P_5$, $P_{11} = P_3P_4 \cap P_1P_6$
step 8	line P_9P_{11}
step 9	$P_{12} = P_9P_{11} \cap P_3P_5$, $P_{13} = P_9P_{11} \cap P_2P_5$, $\mathbf{P_{14} = (b : 1 : 1) \in P_1P_4}$

The construction up to now is visualised in Figure 3.8. In order to improve its readability we denote points P_i merely by their index i. The exception is done for point P_{14} as this is where the new parameter appears.

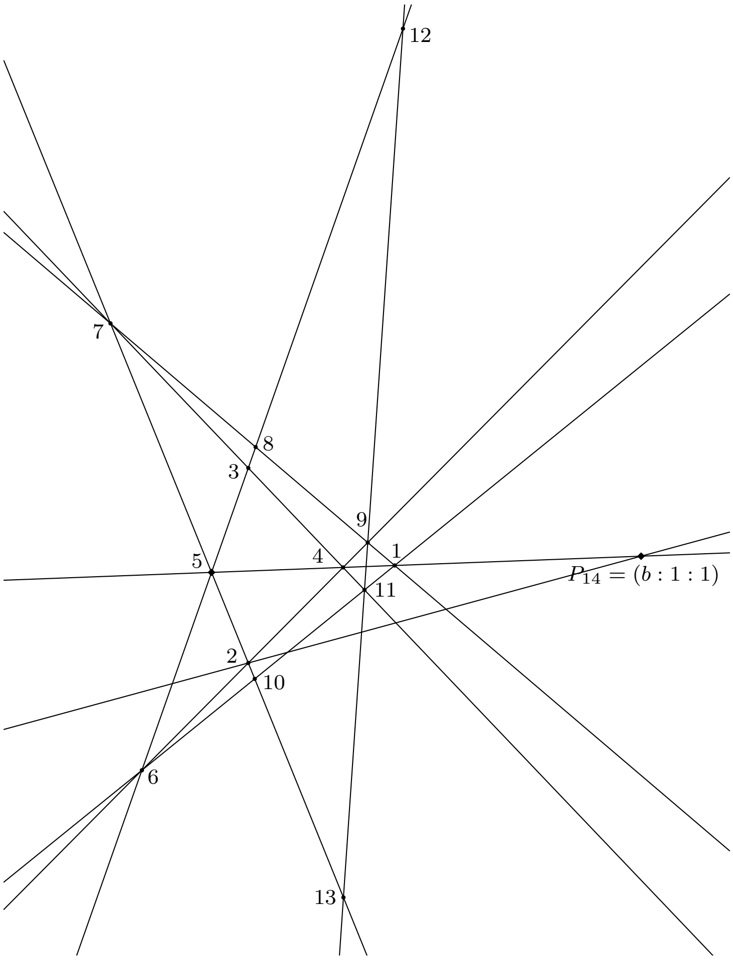

Figure 3.8: The $\mathbb{B}_{16}$ arrangement at the point of choosing the second parameter

step 10	lines: P_2P_{14}, P_3P_{14}
step 11	$P_{15} = P_3P_{14} \cap P_1P_6$, $P_{16} = P_3P_{14} \cap P_9P_{11}$, $P_{17} = P_3P_{14} \cap P_1P_7$, $P_{18} = P_2P_{14} \cap P_3P_4$, $P_{19} = P_2P_{14} \cap P_1P_6$, $P_{20} = P_2P_{14} \cap P_9P_{11}$ $P_{21} = P_2P_{14} \cap P_1P_7$, $P_{22} = P_2P_4 \cap P_3P_{14}$
step 12	lines: P_8P_{20}, $P_{10}P_{16}$, $P_{12}P_{19}$, $P_{13}P_{17}$, $P_{15}P_{18}$, $P_{21}P_{22}$
step 13	$P_{23} = P_3P_{14} \cap P_8P_{20}$, $P_{24} = P_{15}P_{18} \cap P_{13}P_{17}$, $P_{25} = P_{13}P_{17} \cap P_8P_{20}$, $P_{26} = P_{13}P_{17} \cap P_1P_4$, $P_{27} = P_2P_4 \cap P_{15}P_{18}$, $P_{28} = P_{10}P_{16} \cap P_{13}P_{17}$, $P_{29} = P_2P_4 \cap P_{10}P_{16}$, $P_{30} = P_2P_5 \cap P_{12}P_{19}$, $P_{31} = P_{10}P_{16} \cap P_3P_4$, $P_{32} = P_1P_4 \cap P_{15}P_{18}$, $P_{33} = P_1P_6 \cap P_8P_{20}$, $P_{34} = P_1P_7 \cap P_{10}P_{16}$, $P_{35} = P_1P_4 \cap P_{10}P_{16}$

The complete $\mathbb{B}_{16}$ arrangement is visualized in Figure 3.9.

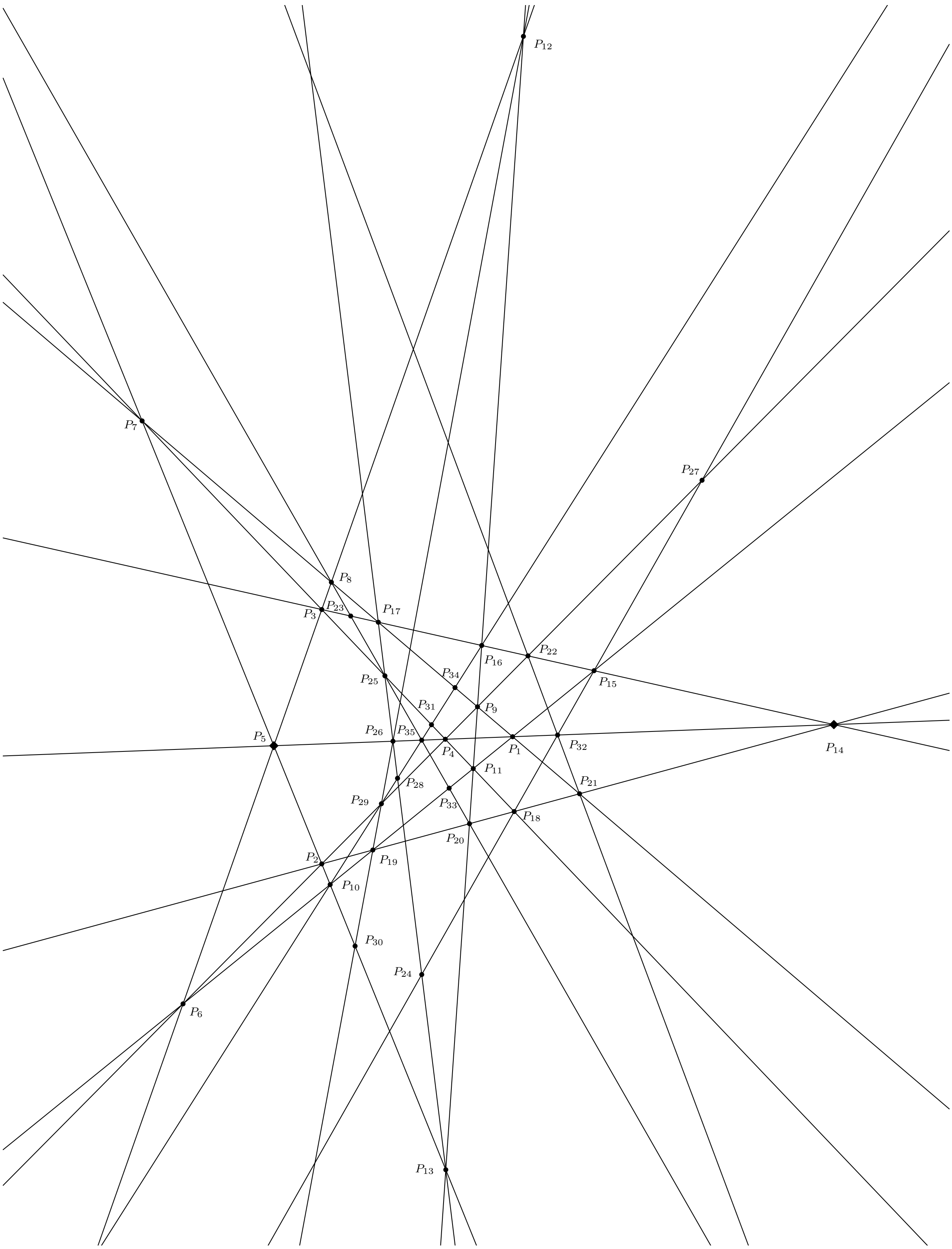

Figure 3.9: The complete $\mathbb{B}_{16}$ arrangement

To sum up, we collect coordinates of all points and equations of all lines in the Tables 3.3, 3.4 and 3.5.

Point	Coordinates
P_1	$(1 : 0 : 0)$
P_2	$(0 : 1 : 0)$
P_3	$(0 : 0 : 1)$
P_4	$(1 : 1 : 1)$
P_5	$(a : 1 : 1)$
P_6	$(a : 1 : a)$
P_7	$(-a : -a : -1)$
P_8	$(-a^2 : -a : -1)$
P_9	$(1 : a : 1)$
P_{10}	$(a^2 : 1 : a)$
P_{11}	$(1 : 1 : a)$
P_{12}	$(a^2 - a : a - 1 : a^3 - 2a + 1)$
P_{13}	$(a^2 - a : a^3 - 2a + 1 : a - 1)$
P_{14}	$(b : 1 : 1)$
P_{15}	$(b : 1 : a)$
P_{16}	$(-ab + b : -a + 1 : -a^2b + a + b - 1)$
P_{17}	$(-ab : -a : -1)$
P_{18}	$(b : b : 1)$
P_{19}	$(-ab : -1 : -a)$
P_{20}	$(-ab + b : -a^2b + a + b - 1 : -a + 1)$
P_{21}	$(b : a : 1)$
P_{22}	$(b : 1 : b)$
P_{23}	$\big(a^4b^2 - a^3b - 2a^2b^2 + a^2b + ab^2 : a^4b - a^3 - 2a^2b + a^2 + ab : a^3 + a^2b^2 - a^2b - a^2 - ab - b^2 + 2b\big)$
P_{24}	$(a^5b^2 - a^4b^2 - a^4b - 2a^3b^2 + 2a^3b + a^2b^3 + a^2b^2 - ab^3 + ab^2 - ab :$
	$a^5b^2 - 2a^4b - 3a^3b^2 + 2a^3b + a^3 + 2a^2b^2 + a^2b - a^2 + ab^2 - 2ab - b^2 + b :$
	$a^4b + a^3b^2 - 3a^3b - a^2b^2 + a^2 + 3ab - a - b)$

Table 3.3: Points in the $\mathbb{B}_{16}$ arrangement, Part 1

Point	Coordinates
P_{25}	$(a^7b - a^6b^2 - a^6 - 2a^5b + a^5 + 3a^4b^2 + a^4 - a^3b^2 + a^3b - a^3 - 2a^2b^2 + ab^2 :$
	$a^7b - a^6b^2 - a^6 - 2a^5b + a^5 + 3a^4b^2 + a^4 - a^3b^2 + a^3b - a^3 - 2a^2b^2 + ab^2 :$
	$a^6 - 2a^5 - a^4b^2 + a^4b - a^4 + a^3b^2 + 3a^3 + a^2b^2 - 2a^2b - ab^2 - a + b)$
P_{26}	$(a^4b - a^3 - 3a^2b + 2a^2 + 2ab - a : a^3 - a^2 - a + 1 : a^3 - a^2 - a + 1)$
P_{27}	$(ab - b : ab - b^2 + b - 1 : ab - b)$
P_{28}	$(a^8b^2 - 2a^7b - 4a^6b^2 + 2a^6b + a^6 + 2a^5b^2 + 3a^5b - a^5 + 4a^4b^2 - 4a^4b - a^4 - 3a^3b^2 + a^3 - a^2b^2 + 2a^2b + ab^2 - ab :$
	$a^6b^2 - a^6b + a^6 - a^5b - a^5 - 3a^4b^2 + 3a^4b - a^4 + a^3b^2 + 2a^3b + a^3 + 2a^2b^2 - 3a^2b - ab^2 - ab + b :$
	$a^7b - a^6b - a^6 - 3a^5b + 2a^5 + a^4b^2 + 2a^4b + a^4 - a^3b^2 + 3a^3b - 3a^3 - a^2b^2 - a^2b + ab^2 - ab + a)$
P_{29}	$(a^4b - a^3 - 2a^2b + a^2 + ab : a^3 + a^2b - 2a^2 - ab + 1 : a^4b - a^3 - 2a^2b + a^2 + ab)$
P_{30}	$(-a^5b + a^4 + 2a^3b - a^3 - a^2b : -a^4 + a^3 - a^2b + 2a^2 + ab - 2a : -a^4b + a^3 + 2a^2b - a^2 - ab)$
P_{31}	$(a^3 - a^2 - ab + b : a^3 - a^2 - ab + b : a^4b - a^3 - 3a^2b + 2a^2 + ab + b - 1)$
P_{32}	$(-ab - b^2 + 2b : -ab + 1 : -ab + 1)$
P_{33}	$(-a^5b + a^4 + 2a^3b - a^2b - a^2 - ab + b : -a^2b + a^2 + b - 1 : -a^3b + a^3 + ab - a)$
P_{34}	$(a^5b - a^4 - 2a^3b + a^2b + a^2 + ab - b : a^3b - a^3 - ab + a : a^2b - a^2 - b + 1)$
P_{35}	$(-a^4b + 2a^3 + 2a^2b - 2a^2 - 2ab + b : -a^2b + a^2 + b - 1 : -a^2b + a^2 + b - 1)$

Table 3.4: Points in the $\mathbb{B}_{16}$ arrangement, Part 2

$$
\begin{aligned}
P_1P_4 &: \quad -y + z = 0\\
P_2P_4 &: \quad x - z = 0\\
P_3P_4 &: \quad -x + y = 0\\
P_2P_5 &: \quad x + (-a)z = 0\\
P_3P_5 &: \quad -x + (a)y = 0\\
P_1P_6 &: \quad (-a)y + z = 0\\
P_1P_7 &: \quad y + (-a)z = 0\\
P_9P_{11} &: \quad (a^2 - 1)x + (-a + 1)y + (-a + 1)z = 0\\
P_2P_{14} &: \quad x + (-b)z = 0\\
P_3P_{14} &: \quad -x + y = 0\\
P_8P_{20} &: \quad (-a^2b + a^2 + b - 1)x + (-a^3 + a^2 + ab - b)y + (a^4b - a^3 - 2a^2b + a^2 + ab)z = 0\\
P_{10}P_{16} &: \quad (-a^2b + a^2 + b - 1)x + (a^4b - a^3 - 2a^2b + a^2 + ab)y + (-a^3 + a^2 + ab - b)z = 0\\
P_{12}P_{19} &: \quad (a^3 - a^2 - a + 1)x + (-a^4b + a^3 + 2a^2b - a^2 - ab)y + (a^2b - a^2 - ab + a)z = 0\\
P_{13}P_{17} &: \quad (-a^3 + a^2 + a - 1)x + (-a^2b + a^2 + ab - a)y + (a^4b - a^3 - 2a^2b + a^2 + ab)z = 0\\
P_{15}P_{18} &: \quad (-ab + 1)x + (ab - b)y + (b^2 - b)z = 0\\
P_{21}P_{22} &: \quad (ab - 1)x + (-b^2 + b)y + (-ab + b)z = 0
\end{aligned}
$$

Table 3.5: Lines in the $\mathbb{B}_{16}$ arrangement

We get the following additional incidences for free: $P_{25} \in P_3P_4$, $P_{26} \in P_{12}P_{19}$, $P_{29} \in P_{12}P_{19}$, $P_{32} \in P_{21}P_{22}$, $P_{35} \in P_8P_{20}$.

Additional incidences impose conditions on parameters a and b. These are the points P_{23},

P_{24}, P_{27}, P_{28}, P_{30}, P_{31}, P_{33} and P_{34}. The conditions to have them triple are

$$\begin{aligned}
P_{23} &= P_3P_{14} \cap P_8P_{20} \cap P_{12}P_{19}, \\
P_{24} &= P_3P_5 \cap P_{13}P_{17} \cap P_{15}P_{18}, \\
P_{27} &= P_2P_4 \cap P_8P_{20} \cap P_{15}P_{18}, \\
P_{28} &= P_2P_{14} \cap P_{10}P_{16} \cap P_{13}P_{17}, \\
P_{30} &= P_2P_5 \cap P_{12}P_{19} \cap P_{21}P_{22}, \\
P_{31} &= P_3P_4 \cap P_{10}P_{16} \cap P_{21}P_{22}, \\
P_{33} &= P_1P_6 \cap P_8P_{20} \cap P_{21}P_{22}, \\
P_{34} &= P_1P_7 \cap P_{10}P_{16} \cap P_{15}P_{18}.
\end{aligned}$$

Evaluating these conditions we get the equation:

$$a^4b^2 - 2a^3b - 2a^2b^2 + a^2b + 2ab^2 - b^3 + a^2 + 2b^2 - 2b = 0$$

Parameter space of configurations $\mathbb{B}_{16}$

The parametrizing curve is

$$C_{16}(a,b) : a^4b^2 - 2a^3b - 2a^2b^2 + a^2b + 2ab^2 - b^3 + a^2 + 2b^2 - 2b = 0.$$

Computations with MAGMA returned genus of $C_{16} = 2$. This curve is hyperelliptic and its Mordell-Weil rank of the Jacobian J has rank 0. Hence, Chabauty's method may be applied and MAGMA computes that the rational points on C_{16} are:

$$(1:1:1),\ (0:1:0),\ (0:0:1),\ (1:0:0),\ (-2:-2:1),\ (-1:1:1),\ (-1:-1:1).$$

All these points lead to degenerate configurations.

3.2.3 Construction of $\mathbb{B}_{18}$

The construction of $\mathbb{B}_{18}$ is also based on the four fundamental points on the projective plane. As before, we introduce parameters a and b satisfying the conditions $a \neq 1$, $b \neq 1$ and $a \neq b$.

step 1	$P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$, $P_4 = (1:1:1)$
step 2	lines: P_1P_4, P_2P_4, P_3P_4
step 3	$\mathbf{P_5 = (a:1:1) \in P_1P_4}$

The construction up to now is visualised at Figure 3.10.

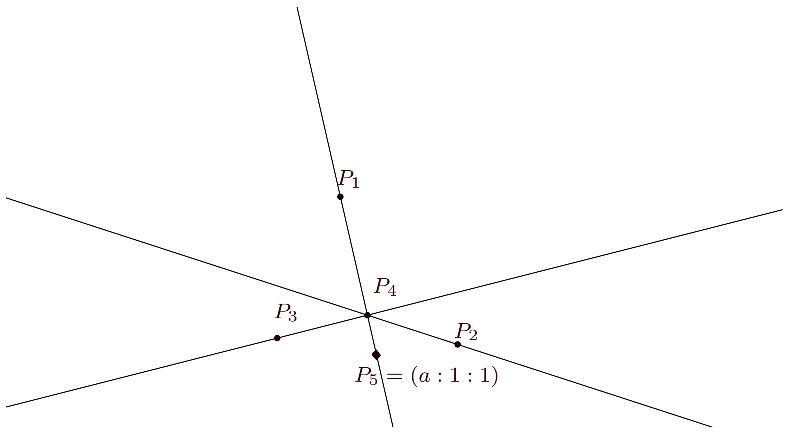

Figure 3.10: The $\mathbb{B}_{18}$ arrangement at the point of choosing the first parameter

step 4	lines: $P_2P_5,\ P_3P_5$
step 5	$P_6 = P_2P_4 \cap P_3P_5,\ P_7 = P_3P_4 \cap P_2P_5$
step 6	$\mathbf{P_8 = (b:1:1) \in P_1P_4}$

The construction up to now is visualised at Figure 3.11.

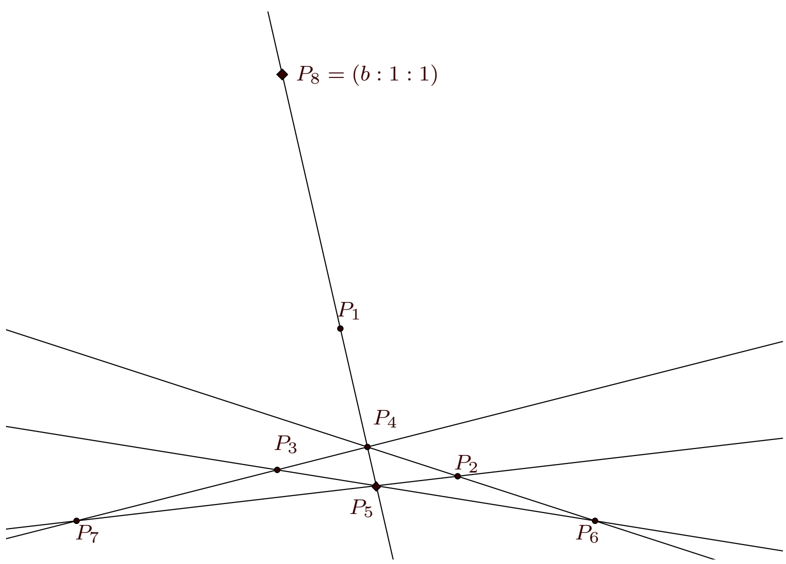

Figure 3.11: The $\mathbb{B}_{18}$ arrangement at the point of choosing the second parameter

step 7	lines: P_6P_8, P_7P_8
step 8	$P_9 = P_2P_5 \cap P_6P_8$, $P_{10} = P_3P_5 \cap P_7P_8$, $P_{11} = P_3P_4 \cap P_6P_8$, $P_{12} = P_2P_4 \cap P_7P_8$
step 9	lines P_1P_9, P_1P_{10}
step 10	$P_{13} = P_3P_4 \cap P_1P_9$, $P_{14} = P_2P_4 \cap P_1P_{10}$, $P_{15} = P_1P_{10} \cap P_6P_8$, $P_{16} = P_1P_9 \cap P_7P_8$, $P_{17} = P_2P_4 \cap P_1P_9$, $P_{18} = P_3P_4 \cap P_1P_{10}$
step 11	lines: $P_{13}P_{15}$, $P_{14}P_{16}$
step 12	$P_{19} = P_2P_4 \cap P_{13}P_{15}$, $P_{20} = P_3P_4 \cap P_{14}P_{16}$, $P_{21} = P_2P_5 \cap P_{13}P_{15}$, $P_{22} = P_3P_5 \cap P_{14}P_{16}$, $P_{23} = P_1P_4 \cap P_{13}P_{15}$, $P_{24} = P_7P_8 \cap P_{13}P_{15}$ $P_{25} = P_6P_8 \cap P_{14}P_{16}$
step 13	lines: $P_{17}P_{21}$, $P_{18}P_{22}$, $P_{19}P_{20}$
step 14	$P_{26} = P_{17}P_{21} \cap P_1P_4$, $P_{27} = P_3P_5 \cap P_{17}P_{21}$, $P_{28} = P_{18}P_{22} \cap P_2P_5$, $P_{29} = P_{19}P_{20} \cap P_{17}P_{21}$, $P_{30} = P_{19}P_{20} \cap P_{18}P_{22}$, $P_{31} = P_3P_4 \cap P_{17}P_{21}$, $P_{32} = P_2P_4 \cap P_{18}P_{22}$, $P_{33} = P_1P_{10} \cap P_{17}P_{21}$, $P_{34} = P_1P_9 \cap P_{18}P_{22}$, $P_{35} = P_{17}P_{21} \cap P_{14}P_{16}$, $P_{36} = P_{18}P_{22} \cap P_{13}P_{15}$, $P_{37} = P_{19}P_{20} \cap P_1P_9$, $P_{38} = P_{19}P_{20} \cap P_1P_{10}$, $P_{39} = P_3P_5 \cap P_{19}P_{20}$, $P_{40} = P_2P_5 \cap P_{19}P_{20}$
step 15	lines: P_2P_{24}, P_3P_{25}, $P_{11}P_{27}$, $P_{12}P_{28}$
step 14	$P_{41} = P_2P_{24} \cap P_1P_{10}$, $P_{42} = P_3P_{25} \cap P_1P_9$, $P_{43} = P_1P_4 \cap P_{11}P_{27}$, $P_{44} = P_2P_{24} \cap P_6P_8$, $P_{45} = P_3P_{25} \cap P_7P_8$, $P_{46} = P_1P_4 \cap P_2P_{24}$

The complete $\mathbb{B}_{18}$ arrangement is visualized in Figure 3.12.

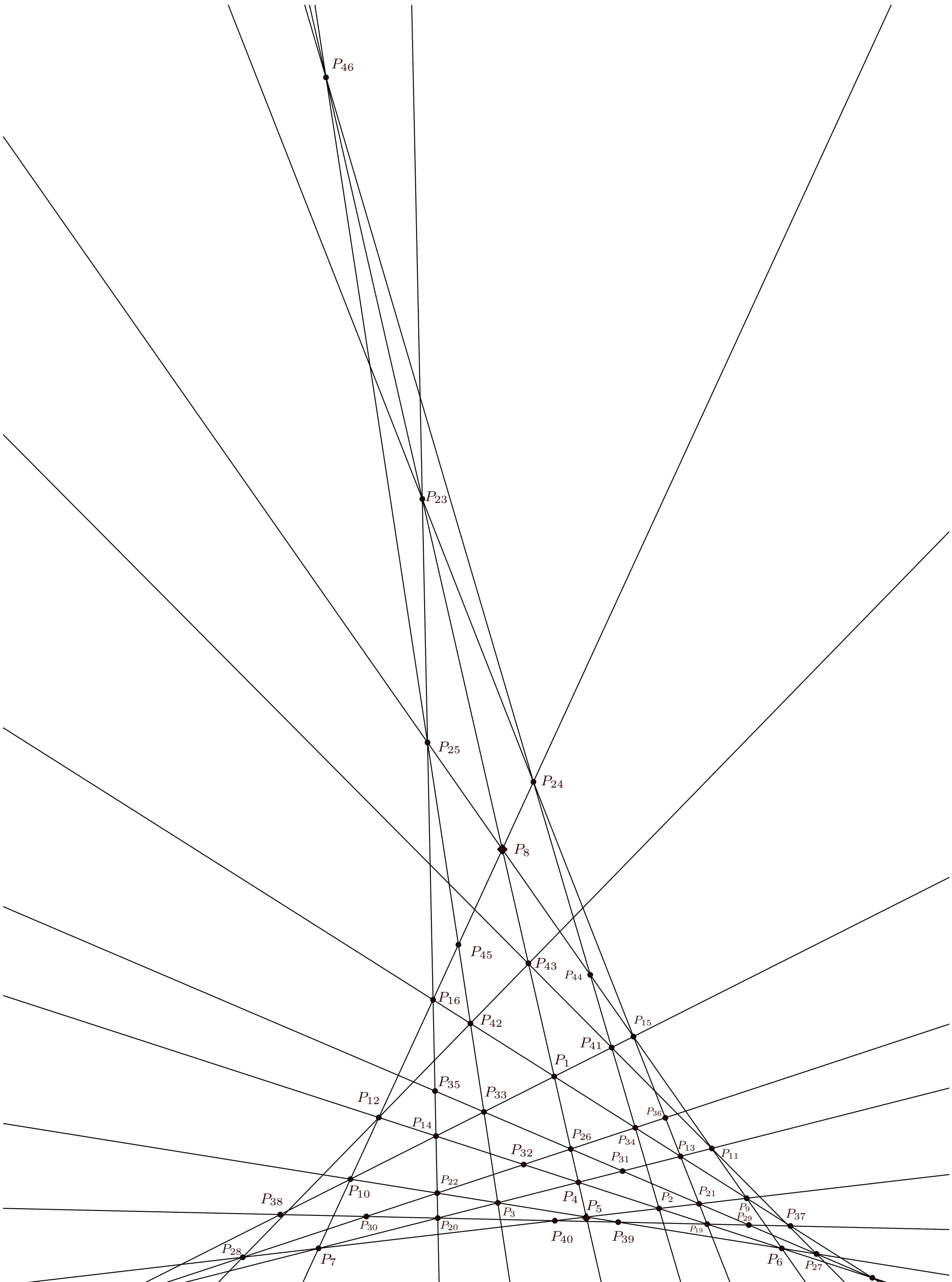

Figure 3.12: The complete $\mathbb{B}_{18}$ arrangement

Thus we obtain 46 triple intersection points. Here there also exist some points which are not triple for arbitrary values of a and b. The conditions that there are 3 lines passing through all of them are:

$$\begin{aligned}
P_{29} &= P_6P_8 \cap P_{17}P_{21} \cap P_{19}P_{20},\\
P_{30} &= P_7P_8 \cap P_{18}P_{22} \cap P_{19}P_{20},\\
P_{31} &= P_2P_{24} \cap P_3P_4 \cap P_{17}P_{21},\\
P_{32} &= P_2P_4 \cap P_3P_{25} \cap P_{18}P_{22},\\
P_{35} &= P_{11}P_{27} \cap P_{12}P_{28} \cap P_{14}P_{16},\\
P_{36} &= P_{11}P_{27} \cap P_{13}P_{15} \cap P_{18}P_{22},\\
P_{37} &= P_1P_9 \cap P_{11}P_{27} \cap P_{19}P_{20},\\
P_{38} &= P_1P_{10} \cap P_{12}P_{28} \cap P_{19}P_{20},\\
P_{39} &= P_2P_{24} \cap P_3P_5 \cap P_{19}P_{20},\\
P_{40} &= P_2P_5 \cap P_3P_{25} \cap P_{19}P_{20},\\
P_{44} &= P_2P_{24} \cap P_6P_8 \cap P_{12}P_{28},\\
P_{45} &= P_3P_{25} \cap P_7P_8 \cap P_{11}P_{27}.
\end{aligned}$$

They imply several algebraic conditions for parameters a and b but only one of them does not lead to a degenerate case. This condition provides us the parametrization curve of the $\mathbb{B}_{18}$ configurations:

$$\begin{aligned}
C_{18}(a,b) :=& a^3b^5 - a^5b^2 + a^4b^3 - 6a^3b^4 + a^2b^5 + a^6 + a^4b^2 + 12a^3b^3 - 4a^2b^4 - 5a^5+\\
& 7a^4b - 22a^3b^2 + 11a^2b^3 - ab^4 + 6a^4 - a^3b + 3a^2b^2 - 4ab^3 + b^4 - 4a^3+\\
& 4a^2b - ab^2.
\end{aligned}$$

Parameter space of configurations $\mathbb{B}_{18}$

Computations with MAGMA returned genus of C_{18} equal to 2 again. This curve is hyperelliptic and the Mordell-Weil rank of its Jacobian J has rank 0. Thus, Chabauty's method may be applied and Magma computes the rational points on homogenization of C_{18}, namely

$$(1:1:1),\ (0:1:0),\ (0:0:1),\ (1:0:0).$$

All these points correspond to degenerate configurations.

3.2.4 Construction of $\mathbb{B}_{24}$

We start the construction by taking the four fundamental points. Then we introduce the parameters a and b, such that $a \neq 1$, $a \neq 0$, $b \neq 1$ and $a \neq b$ and we have

step 1	$P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$, $P_4 = (1:1:1)$
step 2	lines: P_1P_4, P_2P_4, P_3P_4
step 3	$\mathbf{P_5 = (a:1:1) \in P_1P_4}$
step 4	lines: P_2P_5, P_3P_5
step 5	$P_6 = P_2P_4 \cap P_3P_5$, $P_7 = P_2P_5 \cap P_3P_4$
step 6	lines: P_1P_6, P_1P_7
step 7	$P_8 = P_1P_6 \cap P_3P_4$, $P_9 = P_1P_7 \cap P_2P_4$
step 8	lines: P_2P_8, P_3P_9
step 9	$P_{10} = P_1P_4 \cap P_3P_9$, $P_{11} = P_2P_5 \cap P_3P_9$, $P_{12} = P_2P_8 \cap P_3P_5$, $P_{13} = P_1P_6 \cap P_2P_5$, $P_{14} = P_1P_7 \cap P_3P_5$, $P_{15} = P_1P_6 \cap P_3P_9$, $P_{16} = P_1P_7 \cap P_2P_8$
step 10	$\mathbf{P_{17} = (b:1:1) \in P_1P_4}$
step 11	lines: $P_{11}P_{17}$, $P_{12}P_{17}$
step 12	$P_{18} = P_3P_9 \cap P_{12}P_{17}$, $P_{19} = P_2P_8 \cap P_{11}P_{17}$, $P_{20} = P_1P_7 \cap P_{11}P_{17}$, $P_{21} = P_1P_6 \cap P_{12}P_{17}$, $P_{22} = P_3P_4 \cap P_{12}P_{17}$, $P_{23} = P_2P_4 \cap P_{11}P_{17}$, $P_{24} = P_2P_5 \cap P_{12}P_{17}$, $P_{25} = P_3P_5 \cap P_{11}P_{17}$, $P_{26} = P_3P_4 \cap P_{11}P_{17}$, $P_{27} = P_2P_4 \cap P_{12}P_{17}$
step 13	lines: $P_{12}P_{22}$, $P_{22}P_{23}$
step 14	$P_{28} = P_4P_9 \cap P_{22}P_{23}$, $P_{29} = P_{12}P_{17} \cap P_{22}P_{23}$, $P_{30} = P_2P_5 \cap P_{22}P_{23}$, $P_{31} = P_3P_4 \cap P_{22}P_{23}$, $P_{32} = P_1P_{45} \cap P_{22}P_{23}$, $P_{33} = P_1P_6 \cap P_{22}P_{23}$, $P_{34} = P_1P_7 \cap P_{15}P_{22}$, $P_{35} = P_2P_4 \cap P_{15}P_{22}$, $P_{36} = P_3P_5 \cap P_{15}P_{22}$, $P_{37} = P_{11}P_{17} \cap P_{15}P_{22}$, $P_{38} = P_2P_8 \cap P_{15}P_{22}$
step 15	lines: $P_{24}P_{28}$, $P_{25}P_{38}$
step 16	$P_{39} = P_1P_4 \cap P_{24}P_{28}$, $P_{40} = P_2P_4 \cap P_{24}P_{28}$, $P_{41} = P_1P_7 \cap P_{24}P_{28}$, $P_{42} = P_2P_8 \cap P_{24}P_{28}$, $P_{43} = P_3P_4 \cap P_{24}P_{28}$, $P_{44} = P_{15}P_{22} \cap P_{24}P_{28}$, $P_{45} = P_1P_6 \cap P_{24}P_{28}$, $P_{46} = P_3P_5 \cap P_{24}P_{28}$, $P_{47} = P_2P_5 \cap P_{25}P_{38}$, $P_{48} = P_1P_7 \cap P_{25}P_{38}$, $P_{49} = P_{22}P_{23} \cap P_{25}P_{38}$, $P_{50} = P_2P_4 \cap P_{25}P_{38}$, $P_{51} = P_3P_9 \cap P_{25}P_{38}$, $P_{52} = P_1P_6 \cap P_{25}P_{38}$, $P_{53} = P_3P_4 \cap P_{25}P_{38}$
step 17	lines: $P_{30}P_{53}$, $P_{36}P_{40}$
step 18	$P_{54} = P_3P_9 \cap P_{36}P_{40}$, $P_{55} = P_2P_5 \cap P_{36}P_{40}$, $P_{56} = P_2P_8 \cap P_{36}P_{40}$, $P_{57} = P_{12}P_{17} \cap P_{36}P_{40}$, $P_{58} = P_1P_4 \cap P_{36}P_{40}$, $P_{59} = P_3P_4 \cap P_{36}P_{40}$, $P_{60} = P_{25}P_{38} \cap P_{36}P_{40}$, $P_{61} = P_1P_6 \cap P_{36}P_{40}$, $P_{62} = P_1P_7 \cap P_{30}P_{53}$, $P_{63} = P_{24}P_{28} \cap P_{30}P_{53}$, $P_{64} = P_2P_4 \cap P_{30}P_{53}$, $P_{65} = P_{11}P_{17} \cap P_{30}P_{53}$, $P_{66} = P_3P_9 \cap P_{30}P_{53}$, $P_{67} = P_3P_5 \cap P_{30}P_{53}$, $P_{68} = P_2P_8 \cap P_{30}P_{53}$
step 19	lines: $P_{42}P_{54}$, $P_{51}P_{68}$
step 20	$P_{69} = P_2P_4 \cap P_{42}P_{54}$, $P_{70} = P_{15}P_{22} \cap P_{42}P_{54}$, $P_{71} = P_1P_6 \cap P_{42}P_{54}$, $P_{72} = P_1P_4 \cap P_{42}P_{54}$, $P_{73} = P_3P_5 \cap P_{42}P_{54}$, $P_{74} = P_2P_5 \cap P_{51}P_{68}$, $P_{75} = P_1P_7 \cap P_{51}P_{68}$, $P_{76} = P_{22}P_{23} \cap P_{51}P_{68}$, $P_{77} = P_3P_4 \cap P_{51}P_{68}$
step 21	lines: $P_{26}P_{66}$, $P_{27}P_{56}$
step 22	$P_{78} = P_{11}P_{17} \cap P_{27}P_{56}$, $P_{79} = P_3P_5 \cap P_{27}P_{56}$, $P_{80} = P_1P_4 \cap P_{27}P_{56}$, $P_{81} = P_2P_5 \cap P_{26}P_{66}$, $P_{82} = P_{12}P_{17} \cap P_{26}P_{66}$
step 23	lines: $P_{18}P_{65}$, $P_{19}P_{57}$
step 24	$P_{83} = P_3P_9 \cap P_{19}P_{57}$, $P_{84} = P_2P_8 \cap P_{18}P_{65}$, $P_{85} = P_{18}P_{65} \cap P_{19}P_{57}$
step 25	line: $P_{41}P_{52}$

In this case we obtain 85 intersection points and, as previously, some points are not necessarily triple. The conditions to have them triple are:

$$
\begin{aligned}
P_{29} &= P_{12}P_{17} \cap P_{22}P_{23} \cap P_{42}P_{54},\\
P_{33} &= P_{1}P_{6} \cap P_{18}P_{65} \cap P_{22}P_{23},\\
P_{34} &= P_{1}P_{7} \cap P_{15}P_{22} \cap P_{19}P_{57},\\
P_{37} &= P_{11}P_{17} \cap P_{15}P_{22} \cap P_{51}P_{68},\\
P_{43} &= P_{3}P_{4} \cap P_{19}P_{57} \cap P_{24}P_{28},\\
P_{44} &= P_{15}P_{22} \cap P_{18}P_{65} \cap P_{24}P_{28},\\
P_{46} &= P_{3}P_{5} \cap P_{24}P_{28} \cap P_{51}P_{68},\\
P_{47} &= P_{2}P_{5} \cap P_{25}P_{38} \cap P_{42}P_{54},\\
P_{49} &= P_{22}P_{23} \cap P_{19}P_{57} \cap P_{25}P_{38},\\
P_{50} &= P_{2}P_{4} \cap P_{18}P_{65} \cap P_{25}P_{38},\\
P_{55} &= P_{2}P_{5} \cap P_{36}P_{40} \cap P_{41}P_{52},\\
P_{59} &= P_{3}P_{4} \cap P_{18}P_{65} \cap P_{36}P_{40},\\
P_{60} &= P_{25}P_{38} \cap P_{26}P_{66} \cap P_{36}P_{40},\\
P_{61} &= P_{1}P_{6} \cap P_{36}P_{40} \cap P_{51}P_{68},\\
P_{62} &= P_{1}P_{7} \cap P_{30}P_{53} \cap P_{42}P_{54},\\
P_{63} &= P_{24}P_{28} \cap P_{27}P_{56} \cap P_{30}P_{53},
\end{aligned}
$$

$$
\begin{aligned}
P_{64} &= P_{2}P_{4} \cap P_{19}P_{57} \cap P_{30}P_{53},\\
P_{67} &= P_{3}P_{5} \cap P_{30}P_{53} \cap P_{41}P_{52},\\
P_{69} &= P_{2}P_{4} \cap P_{41}P_{52} \cap P_{42}P_{54},\\
P_{71} &= P_{1}P_{6} \cap P_{19}P_{57} \cap P_{42}P_{54},\\
P_{73} &= P_{3}P_{5} \cap P_{18}P_{65} \cap P_{42}P_{54},\\
P_{74} &= P_{2}P_{5} \cap P_{19}P_{57} \cap P_{51}P_{68},\\
P_{75} &= P_{1}P_{7} \cap P_{18}P_{65} \cap P_{51}P_{68},\\
P_{76} &= P_{22}P_{23} \cap P_{26}P_{66} \cap P_{51}P_{68},\\
P_{77} &= P_{3}P_{4} \cap P_{41}P_{52} \cap P_{51}P_{68},\\
P_{78} &= P_{11}P_{17} \cap P_{27}P_{56} \cap P_{41}P_{52},\\
P_{79} &= P_{3}P_{5} \cap P_{19}P_{57} \cap P_{27}P_{56},\\
P_{81} &= P_{2}P_{5} \cap P_{18}P_{65} \cap P_{26}P_{66},\\
P_{82} &= P_{12}P_{17} \cap P_{26}P_{66} \cap P_{41}P_{52},\\
P_{83} &= P_{3}P_{9} \cap P_{19}P_{57} \cap P_{41}P_{52},\\
P_{84} &= P_{2}P_{8} \cap P_{18}P_{65} \cap P_{41}P_{52}.
\end{aligned}
$$

Removing the factors which lead to the degenerate cases we are left with the equation:

$$
\begin{aligned}
C_{24}(a,b) :=& a^8b^3 + a^7b^3 + a^6b^4 - 6a^7b^2 + 3a^6b^3 - 6a^6b^2 - 2a^5b^3 + 10a^6b - 6a^5b^2 -\\
& 2a^4b^3 - a^6 + 12a^5b + 3a^4b^2 - 2a^3b^3 - 6a^5 + 3a^4b + 6a^3b^2 - a^2b^3 -\\
& 3a^4 - 13a^3b + 9a^2b^2 - ab^3 + 4a^3 - 12a^2b + 6ab^2 - b^3 + 5a^2 - 3ab +\\
& 2a - b = 0.
\end{aligned}
$$

This condition is necessary for the construction to terminate successfully in the sense that we obtain exactly 85 triple points on 24 lines satisfying the combinatorial properties of the original Böröczky configuration of 24 lines.

Parameter space of configurations $\mathbb{B}_{24}$

The parametrizing curve

$$\begin{aligned}C_{24}(a,b) :=& a^8b^3 + a^7b^3 + a^6b^4 - 6a^7b^2 + 3a^6b^3 - 6a^6b^2 - 2a^5b^3 + 10a^6b - 6a^5b^2 - \\ & 2a^4b^3 - a^6 + 12a^5b + 3a^4b^2 - 2a^3b^3 - 6a^5 + 3a^4b + 6a^3b^2 - a^2b^3 - \\ & 3a^4 - 13a^3b + 9a^2b^2 - ab^3 + 4a^3 - 12a^2b + 6ab^2 - b^3 + 5a^2 - 3ab + \\ & 2a - b = 0\end{aligned}$$

is a curve of genus 5. Therefore, there are no general methods to determine all the rational points of C_{24}. The only rational points of height up to 10^5 are

$$(1:1:1),\quad (0:1:0),\quad (0:0:1),\quad (1:0:0),\quad (-1:-1:1).$$

Simple but cumbersome calculations confirm that all of these points lead to a degenerate configuration, however we were not able to prove that the construction cannot be made over the rational numbers.

Chapter 4

Line arrangements and the Containment Problem

Before we present our main results, let us give a chronological outline of the subject in order to present a motivation standing behind our work. Here we assume that $\mathcal{P} \subset \mathbb{P}^2_{\mathbb{C}}$ is a finite set of mutually distinct points, and we denote by I its radical ideal.

(**2001**) Ein, Lazarsfeld, and Smith [16]: $I^{(2k)} \subset I^k$ for every $k \geq 1$.

(**2006**) Huneke: Does the containment $I^{(3)} \subset I^2$ hold?

(**2009**) Bocci and Harbourne: Does the containment $I^{(2k-1)} \subset I^k$ hold for every $k \geq 1$?

(**2013**) Dumnicki, Szemberg, and Tutaj-Gasińska [15]: The first counterexample to the containment $I^{(3)} \subset I^2$ – they used the dual-Hesse arrangement of 9 lines and 12 triple intersection points.

(**2013**) Czapliński *et al.* [7]: The first counterexample to the containment $I^{(3)} \subset I^2$ over the real numbers – Böröczky's arrangement of 12 lines, 19 triple and 9 double intersection points.

(**2015**) Lampa-Baczyńska and Szpond [27]: The first counterexample to the containment $I^{(3)} \subset I^2$ over the rational numbers – using the parameter space of Böröczky arrangement of 12 lines they found a rational realization of this combinatorics.

(**2015**) Harbourne: Construct new counterexamples to the containment $I^{(3)} \subset I^2$ over the rational numbers using parameter spaces of Böröczky's line arrangements.

(**2018**) Ma and Schwede [29]: $I^{(2k)} \subset I^k$ for every $k \geq 1$ in the mixed characteristic case.

4.1 Containment criteria

Over the years a number of containment criteria has been developed. We recall here those which are relevant for our applications. We begin by recalling some standard notions in a general setting of homogeneous ideals in the ring of polynomials. In order to fix the notation, let I be a homogeneous ideal in the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_n]$. Let

$$0 \to \ldots \to \oplus_j R(-j)^{\beta_{i,j}(I)} \to \ldots \to \oplus_j R(-j)^{\beta_{1,j}(I)} \to \oplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$

be the minimal free resolution of I. From this resolution we derive one of central invariants in commutative algebra and algebraic geometry.

Definition 4.1.1. The Castelnuovo-Mumford regularity (or simply, regularity) of I, denoted by $\operatorname{reg}(I)$, is the integer

$$\operatorname{reg}(I) = \max\left\{j - i : \ \beta_{i,j}(I) \neq 0\right\}.$$

Thus $\operatorname{reg}(I)$ is the height of the Betti table of I.

Another important invariant of a homogeneous ideal $I = \oplus_{t=0}^{\infty}(I)_t$ is its initial degree

$$\alpha(I) = \min\left\{t : \ (I)_t \neq 0\right\} = \min\left\{j : \ \beta_{0,j} \neq 0\right\}.$$

Note that it is always

$$\alpha(I) \leq \operatorname{reg}(I)$$

because $\operatorname{reg}(I)$ is at least equal to the maximal degree of a generator in the minimal set of generators.

Bocci and Harbourne proved in [3, Lemma 2.3.3 (c)] an important containment statement, which we recall here only in the case of saturated ideals of zero-dimensional subschemes in $\mathbb{P}^n$.

Proposition 4.1.2 (Bocci-Harbourne Containment Criterion)**.** *Let $I \subset R$ be a non-trivial saturated homogeneous ideal defining a zero-dimensional subscheme. For $t \geq r \cdot \operatorname{reg}(I)$ there is*

$$(I^r)_t = (I^{(r)})_t.$$

Remark 4.1.3. It follows from the proof of Lemma 2.3.3 in [3] that the conclusion in Proposition 4.1.2 holds as soon, as $t \geq \operatorname{reg}(I^r)$.

From Proposition 4.1.2 and Remark 4.1.3 we derive the following useful result.

Corollary 4.1.4 (Bocci-Harbourbe Containment Criterion 2). *Let $I \subset R$ be a non-trivial ideal defining a zero-dimensional subscheme in $\mathbb{P}^n$.*

$$\textit{If } \operatorname{reg}(I^r) \leq \alpha(I^{(m)}), \textit{ then } I^{(m)} \subset I^r.$$

In the rest of this section, we consider zero-dimensional strict almost complete intersections, i.e., ideals of height h that have a minimal set of generators of cardinality $h+1$. In the case of projective plane, a reduced set of points is strict almost complete intersection if its ideal is 3-generated – the minimal set of homogeneous generators of degree d has cardinality 3. Let $I = (f, g, h) \subset R := \mathbb{K}[x, y, z]$ (here we do not assume anything about $\mathbb{K}$) be a homogeneous ideal with minimal generators of the same degree. We are interested in free resolutions for powers of I, and in order to do so we need to consider the Rees algebra of I, which is defined by $\mathcal{R}(I) = \oplus_{i \geq 0} I^i t^i$. In that case we have the following description, see [33].

Theorem 4.1.5. *Let I be a strict almost complete intersection ideal defining a reduced set of points in $\mathbb{P}^2$ and let $A = \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \end{pmatrix}^T$ be a presentation matrix for the module of syzygies on I, i.e., the Hilbert-Burch matrix of I. Then the Rees algebra of I is given as a quotient of the polynomial ring $S = R(T_1, T_2, T_3)$ of the following form*

$$\mathcal{R}(I) \cong S/(P_1T_1 + P_2T_2 + P_3T_3, Q_1T_1 + Q_2T_2 + Q_3T_3).$$

Furthermore, the defining ideal of this algebra, $(P_1T_1 + P_2T_2 + P_3T_3, Q_1T_1 + Q_2T_2 + Q_3T_3)$ is a complete intersection.

Before we present our main tool in the box, we need the following result providing a precise description of powers of strict almost complete intersection ideals.

Theorem 4.1.6. *Let I be a strict almost complete intersection ideal with minimal generators of the same degree d defining a reduced set of points in $\mathbb{P}^2_{\mathbb{K}}$. Let $A^T = \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \end{pmatrix}$ be the Hilbert-Burch matrix of I. Let d_0 and d_1 denote the respective degrees of the polynomials in each of the two rows of A^T. Then the minimal free resolutions of I^2 and I^3 are as follows:*

$$0 \to R(-3d) \xrightarrow{X} R(-2d - d_0)^3 \oplus R(-2d - d_1)^3 \longrightarrow R(-2d)^6 \longrightarrow I^2 \longrightarrow 0,$$

$$0 \longrightarrow R(-4d)^3 \xrightarrow{Y} R(-3d-d_0)^6 \oplus R(-3d-d_1)^6 \longrightarrow R(-3d)^{10} \longrightarrow I^3 \longrightarrow 0,$$

and the last homomorphism in the respective resolutions can be described by the matrices X and Y given below by:

$$X = [P_1, \quad P_2, \quad P_3, \quad -Q_1, \quad -Q_2, \quad -Q_3]^T,$$

and

$$Y = \begin{pmatrix} P_1 & P_2 & P_3 & 0 & 0 & 0 & -Q_1 & -Q_2 & -Q_3 & 0 & 0 & 0 \\ 0 & P_1 & 0 & P_2 & P_3 & 0 & 0 & -Q_1 & 0 & -Q_2 & -Q_3 & 0 \\ 0 & 0 & P_1 & 0 & P_2 & P_3 & 0 & 0 & -Q_1 & 0 & -Q_2 & -Q_3 \end{pmatrix}^T.$$

Theorem 4.1.7 (Seceleanu)**.** *Let I be a 3-generated homogeneous ideal with minimal generators f, g, h of the same degree d, defining a reduced set of points in $\mathbb{P}^2_{\mathbb{K}}$, where $\mathbb{K}$ is an arbitrary field of characteristic different than 3. Set Y to be the matrix representing the last homomorphism in the minimal free resolution of I^3 (see above):*

$$0 \longrightarrow R^3 \xrightarrow{Y} R^{12} \longrightarrow R^{10} \longrightarrow I^3 \longrightarrow 0.$$

Then $I^{(3)} \subseteq I^2$ if and only if $[f, g, h]^T \in \mathrm{Image}(Y^T)$.

In the same direction, we can follow ideas of Grifo, Huneke, and Mukundan developed in [21]. In order to formulate more efficient criterion on the containment $I^3 \subset I^2$ for ideals generated by 2×2 minors of 2×3 matrices.

Theorem 4.1.8 (Grifo-Huneke-Mukundan)**.** *Let $R = \mathbb{K}[x, y, z]$, where $\mathbb{K}$ is a field of characteristic different than 3. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in R$ and consider the ideal I which is generated by 2×2 minors of the matrix*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

If the ideal $\langle a_1, a_2, a_3, b_1, b_2, b_3 \rangle$ can be generated by 5 or less elements, than $I^{(3)} \subset I^2$.

It turns out that we can use this interesting result in a straightforward way in the case of Böröczky's arrangements of $n \in \{4, ..., 10\}$ lines in order to verify that for the radical ideals of triple intersection points I_3 the containment $I^{(3)} \subset I^2$ **does** hold. Since the method is the same for all cases, we are going to present our considerations only for $n = 10$.

Proposition 4.1.9. *Let I_3 be the radical ideal of the triple intersection points of Böröczky's arrangement of* 10 *lines. Then the containment $I_3^{(3)} \subset I_3^2$ does hold.*

Proof. First of all, we need to observe that the ideal of the triple intersection points is generated as bellow by

$$I_3 = \langle 4xy^3 + 2x^2yz + 4y^3z - xyz^2 - 3yz^3, 4x^3y + 2x^2yz - 3xyz^2 - yz^3,$$
$$x^4 - 6x^2y^2 + y^4 - 4x^3z + x^2z^2 + y^2z^2 + 2xz^3 - z^4\rangle.$$

Since the ideal I_3 is 3-generated, we can use the theory of Hilbert-Burch. We compute the minimal free resolution of I_3, and the matrix A that we are searching for is given by the following Hilbert-Burch matrix, namely

$$A = \begin{pmatrix} 4x^2 - 2xz - z^2 & 4y^2 - 14xz + z^2 & -4y^2 - 2xz + 3z^2 \\ 4x^2 - 24y^2 - 14xz + 13z^2 & 0 & -16xy - 16yz \end{pmatrix}.$$

Since it is obvious that the ideal given by the entries of matrix A is 5 or less generated (in fact it is 5 generated), thus the containment $I_3^{(3)} \subset I_3^2$ holds. □

Now we are going to consider the last remaining case which would allow us to conclude that the minimal counterexample to the containment problem $I^{(3)} \subset I^2$ (in the sense of the number of lines) for Böröczky's family of line arrangements is the case of 12 lines. As a first observation, we can show that for $n = 11$ lines the ideal of the triple intersection points is not 3-generated – in fact the minimal set of generators has cardinality 4, so we cannot use the Grifo-Huneke-Mukundan method. In the remaining part of this section, we are going to show explicitly the following theorem.

Theorem 4.1.10. *Let us denote by I_3 the radical ideal of the triple intersection points of Böröczky's arrangement of* 11 *lines. Then the containment $I_3^{(3)} \subset I_3^2$ holds.*

Proof. Our proof heavily relies on computer aid methods with use of Singular. First of all, we compute the ideal I_3 which has exactly 4 generators, namely

$$\begin{aligned} I_3 = \langle & 4x^3y - 4xy^3 - 3x^2yz - 3y^3z - 2xyz^2 + 2yz^3, \\ & 32y^5 + 88xy^3z + 33x^2yz^2 - 55y^3z^2 - 66xyz^3 + 22yz^4, \\ & 32x^2y^3 + 72xy^3z + 11x^2yz^2 + 35y^3z^2 - 22xyz^3 - 22yz^4, \\ & 2x^5 - 10xy^4 - 8x^4z - 15x^2y^2z - 7y^4z + 4x^3z^2 + 2xy^2z^2 + 10x^2z^3 + 8y^2z^3 - 4xz^4 - 2z^5\rangle. \end{aligned}$$

Then we compute the minimal free resolution of I_3^2 which has the following form

$$0 \to S(-13)^2 \oplus S(-12) \to S(-12)^3 \oplus S(-11)^7 \oplus S(-10)^2 \to S(-10)^6 \oplus S(-9)^3 \oplus S(-8) \to I_3^2 \to 0.$$

Thus we have $\mathrm{reg}(I_3^2) = 11$. Taking into account Corollary 2.1.9 we obtain $\alpha(I_3^{(3)}) = 11$. Applying in turn Corollary 4.1.4 with $m = 3$ and $r = 2$ we conclude that

$$I_3^{(3)} \subset I_3^2.$$

□

4.2 Arrangements of 12 lines with 19 triple and 9 double intersection points

The study of arrangements of lines with many triple is motivated by the Sylvester-Gallai problem and classical problems in combinatorics, see [34], and for a modern treatment [14]. A general upper bound for the number $T_3(d)$ of triple points in the arrangement of d lines has been provided by Schönheim

$$T_3(d) \leq \left\lfloor \left\lfloor \frac{d-1}{3} \right\rfloor \cdot \frac{d}{3} \right\rfloor - \varepsilon(d),$$

where $\varepsilon(d) = 1$ if $d \equiv 5 \bmod 6$ and $\varepsilon(d) = 0$, otherwise. In the case of 12 lines, the upper bound is 20. It has been proved by Burr, Grünbaum and Sloane [6, Theorem 7] that no arrangement of 12 lines with 20 triple points is possible over the reals. We do not know if it exists over the complex numbers, but we suspect that the answer is **no**. On the other hand, Bokowski and Pokora classified in [5] all oriented matroids of rank 3 with 12 pseudolines and 19 triple points. They showed that among them, there are only three realizable matroids, i.e., corresponding to actual arrangements of lines. In their notation, these arrangements are C_2, C_6 (which is Böröczky on 12 lines), and C_7. In this section we take a closer look at C_2 and C_7. Our study is motivated by the following folklore problem.

Question 4.2.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line arrangements having the same number of lines and the same number of the corresponding t_k points, i.e., the same weak combinatorics. Assume that the containment $I_{\mathcal{L}_1}^{(3)} \subset I_{\mathcal{L}_1}^2$ does not hold. Does it follow that the containment $I_{\mathcal{L}_2}^{(3)} \subset I_{\mathcal{L}_2}^2$ is not satisfied?

We answer this question in the negative. Arrangements of 12 lines are of interest in the context of the containment problem since 12 is the smallest known number d of real lines such that triple points of arrangement of d lines provide a non-containment example for $I^{(3)} \subset I^2$, Interestingly, all arrangements considered here can be realized over the reals, this follows from the Bokowski-Pokora classification.

4.2.1 Arrangement C_2

Figure 4.1 indicates the C_2 arrangement. The points $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$ are at infinity. Without loss of generality, we assume that $P_3 = (0 : 0 : 1)$ and $P_4 = (1 : 1 : 1)$. Now we want to show that, up to conjugation in the field extension $\mathbb{Q}[\sqrt{2}]/\mathbb{Q}$, coordinates of all remaining points are determined by incidences depicted in Figure 4.1.

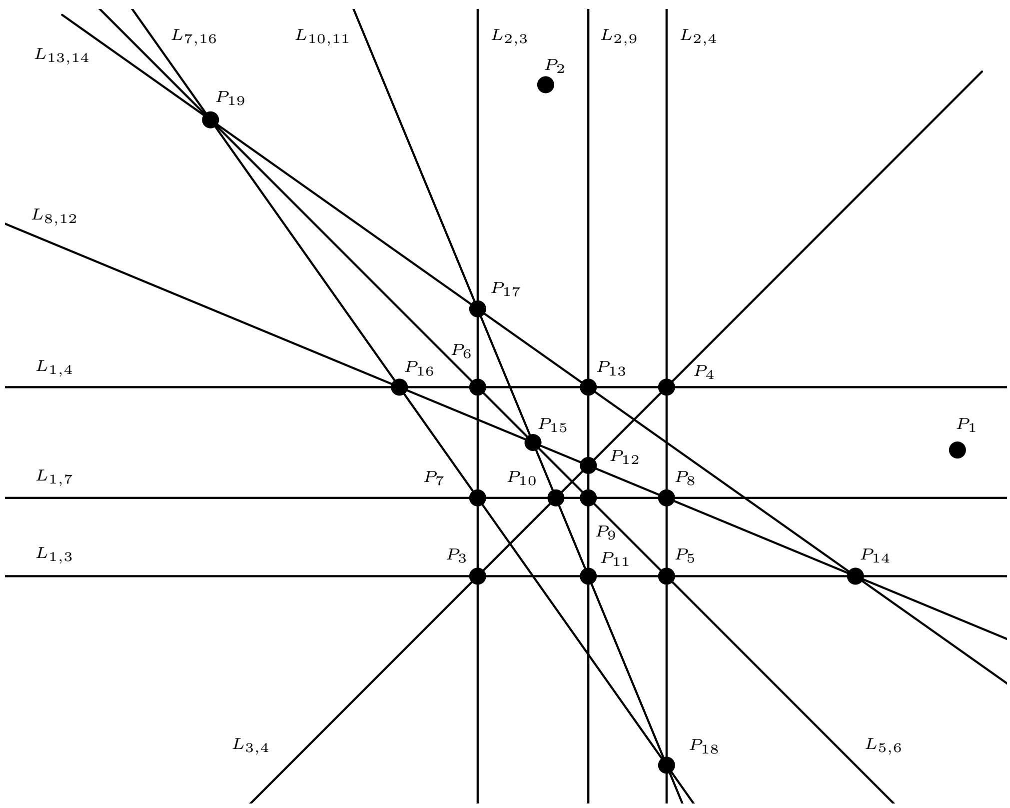

Figure 4.1: Arrangement C_2.

We take the following lines:

$$L_{1,3}:\ y=0,$$
$$L_{2,4}:\ x-z=0,$$
$$L_{1,4}:\ y-z=0,$$
$$L_{3,4}:\ x-y=0,$$
$$L_{2,3}:\ x=0,$$

where $L_{i,j}$, is the line passing through the points P_i and P_j. Then we obtain the points

$$P_5 = L_{1,3}\cap L_{2,4} = (1:0:1),$$
$$P_6 = L_{1,4}\cap L_{2,3} = (0:1:1)$$

and the line

$$L_{5,6}:\ x+y-z=0.$$

We need now to introduce a parameter to proceed with the construction. Thus we take the point $P_7=(0,1,a)\in L_{2,3}$. Since all points and lines in the configuration should be distinct, we assume that $a\neq 1$ and $a\neq 0$. We obtain the remaining lines and points in the following order:

$$L_{1,7}:\ z-ay=0,$$

$$P_8 = L_{1,7}\cap L_{2,4} = (a:1:a),$$
$$P_9 = L_{1,7}\cap L_{5,6} = (a-1:1:a),$$
$$P_{10} = L_{1,7}\cap L_{3,4} = (1:1:a),$$
$$L_{2,9}:\ ax-(a-1)z=0,$$

$$P_{11} = L_{2,9}\cap L_{1,3} = (a-1:0:a),$$
$$P_{12} = L_{2,9}\cap L_{3,4} = (a-1:a-1:a),$$
$$P_{13} = L_{2,9}\cap L_{1,4} = (a-1:a:a),$$
$$L_{8,12}:\ a(2-a)x-ay+(a-1)^2z=0,$$

$$P_{14} = L_{8,12}\cap L_{1,3} = ((a-1)^2:0:a(a-2)),$$
$$P_{15} = L_{8,12}\cap L_{5,6} = (a^2-3a+1:-1:a(a-3)),$$
$$P_{16} = L_{8,12}\cap L_{1,4} = (a^2-3a+1:a(a-2):a(a-2)),$$
$$L_{10,11}:\ ax-a(a-2)y-(a-1)z=0,$$

$$P_{17} = L_{10,11}\cap L_{2,3} = (0:a-1:a(a-2)),$$
$$P_{18} = L_{10,11}\cap L_{2,4} = (a(2-a):1:a(2-a)),$$

$$L_{13,14}: \; a(a-2)x + (a-1)y - (a-1)^2 z = 0,$$

$$L_{7,16}: \; a(a-1)(a-2)x - a(a^2-3a+1)y + (a^2-3a+1)z = 0,$$

$$P_{19} = L_{13,14} \cap L_{7,16} = (a^5 - 5a^4 + 7a^3 - a^2 - 3a + 1 : a^3(a-2)^2 : a^5 - 4a^4 + 3a^3 + 3a^2 - 2a).$$

The following incidences need to be checked additionally:

$$P_{15} \in L_{10,11}, \; P_{17} \in L_{13,14}, \; P_{18} \in L_{7,16}, \; P_{19} \in L_{5,6}.$$

By the determinant condition we conclude that $P_{15} \in L_{10,11}$ holds without any assumption on a, but the remaining incidences occur under the algebraic condition

$$a^2 - 2a - 1 = 0,$$

which is equivalent to $a = 1 + \sqrt{2}$ or $a = 1 - \sqrt{2}$. The arrangement in Figure 4.1 corresponds to the parameter $a = 1 + \sqrt{2}$. Then $P_7 = (0 : \sqrt{2} - 1 : 1)$ and $P_{17} = (0 : \sqrt{2} : 1)$.

We sum up the discussion gathering coordinates of the points and equations of the lines.

Point	Coordinates
P_1	$(1:0:0)$
P_2	$(0:1:0)$
P_3	$(0:0:1)$
P_4	$(1:1:1)$
P_5	$(1:0:1)$
P_6	$(0:1:1)$
P_7	$\left(0:\sqrt{2}-1:1\right)$
P_8	$\left(1:\sqrt{2}-1:1\right)$
P_9	$\left(2-\sqrt{2}:\sqrt{2}-1:1\right)$
P_{10}	$\left(\sqrt{2}-1:\sqrt{2}-1:1\right)$
P_{11}	$\left(2-\sqrt{2}:0:1\right)$
P_{12}	$\left(2-\sqrt{2}:2-\sqrt{2}:1\right)$
P_{13}	$\left(2-\sqrt{2}:1:1\right)$
P_{14}	$(2:0:1)$
P_{15}	$\left(\frac{2-\sqrt{2}}{2}:\frac{\sqrt{2}}{2}:1\right)$

Table 4.1: Points in C_2^+ arrangement

P_{16}	$\left(1-\sqrt{2}:1:1\right)$
P_{17}	$\left(0:\sqrt{2}:1\right)$
P_{18}	$(1:-1:1)$
P_{19}	$\left(-\sqrt{2}:1+\sqrt{2}:1\right)$

Table 4.2: Points in C_2^+ arrangement, continued

$P_1P_3:$	$y=0$
$P_2P_4:$	$x-z=0$
$P_1P_4:$	$y-z=0$
$P_3P_4:$	$x-y=0$
$P_2P_3:$	$x=0$
$P_5P_6:$	$x+y-z=0$
$P_1P_7:$	$z-(1+\sqrt{2})y=0$
$P_2P_9:$	$(1+\sqrt{2})x-\sqrt{2}z=0$
$P_8P_{12}:$	$x+(1+\sqrt{2})y-2z=0$
$P_{10}P_{11}:$	$(1+\sqrt{2})x+y-\sqrt{2}z=0$
$P_{13}P_{14}:$	$(1+\sqrt{2})x+(2+\sqrt{2})y-2(1+\sqrt{2})z=0$
$P_7P_{16}:$	$\sqrt{2}x+y-(\sqrt{2}-1)z=0$

Table 4.3: Lines in the C_2^+ arrangement

Point	Coordinates
P_1	$(1:0:0)$
P_2	$(0:1:0)$
P_3	$(0:0:1)$
P_4	$(1:1:1)$
P_5	$(1:0:1)$
P_6	$(0:1:1)$
P_7	$\left(0:-1-\sqrt{2}:1\right)$

Table 4.4: Points in C_2^- arrangement

P_8	$\left(1 : -1-\sqrt{2} : 1\right)$
P_9	$\left(2+\sqrt{2} : -\sqrt{2}-1 : 1\right)$
P_{10}	$\left(-1-\sqrt{2} : -1-\sqrt{2} : 1\right)$
P_{11}	$\left(2+\sqrt{2} : 0 : 1\right)$
P_{12}	$\left(2+\sqrt{2} : 2+\sqrt{2} : 1\right)$
P_{13}	$\left(2+\sqrt{2} : 1 : 1\right)$
P_{14}	$(2 : 0 : 1)$
P_{15}	$\left(\frac{2+\sqrt{2}}{2} : -\frac{\sqrt{2}}{2} : 1\right)$
P_{16}	$\left(1+\sqrt{2} : 1 : 1\right)$
P_{17}	$\left(0 : -\sqrt{2} : 1\right)$
P_{18}	$(1 : -1 : 1)$
P_{19}	$\left(\sqrt{2} : 1-\sqrt{2} : 1\right)$

Table 4.5: Points in the C_2^- arrangement, continued

$$
\begin{aligned}
P_1P_3 &: \quad y = 0 \\
P_2P_4 &: \quad x - z = 0 \\
P_1P_4 &: \quad y - z = 0 \\
P_3P_4 &: \quad x - y = 0 \\
P_2P_3 &: \quad x = 0 \\
P_5P_6 &: \quad x + y - z = 0 \\
P_1P_7 &: \quad z + (-1+\sqrt{2})y = 0 \\
P_2P_9 &: \quad (-1+\sqrt{2})x - \sqrt{2}z = 0 \\
P_8P_{12} &: \quad x + (1-\sqrt{2})y - 2z = 0 \\
P_{10}P_{11} &: \quad (1-\sqrt{2})x + y + \sqrt{2}z = 0 \\
P_{13}P_{14} &: \quad (1-\sqrt{2})x + (2-\sqrt{2})y + 2(-1+\sqrt{2})z = 0 \\
P_7P_{16} &: \quad \sqrt{2}x - y - (1+\sqrt{2})z = 0
\end{aligned}
$$

Table 4.6: Lines in the C_2^- arrangement

Corollary 4.2.2. *There is no geometric realization of C_2 over the rational numbers possible.*

Remark 4.2.3. We observe that the moduli space for $C_2^{\pm}$ arrangements is similar to the situation for the MacLane arrangements. Namely, the moduli space consists of two points and it is not irreducible, in particular.

Now we are in the position to prove our main statement for the ideal I of triple points in the C_2 arrangement.

Proposition 4.2.4. *Let I be the ideal of triple points in C_2 arrangement, then*

$$I^{(3)} \subset I^2.$$

Proof. Using Singular, we check that $\text{reg}(I) = 6$. Then $\text{reg}(I^2) \leq 12$. On the other hand, by Corollary 2.1.9 we have $\alpha(I^{(3)}) = 12$. Combining this with result 4.1.4, Bocci-Harbourne's Criterion 2, we obtain the assertion. □

4.2.2 Arrangement C_7

The real realization of the configuration C_7 is in the picture below (points P_1, P_2, P_3 are "at infinity"):

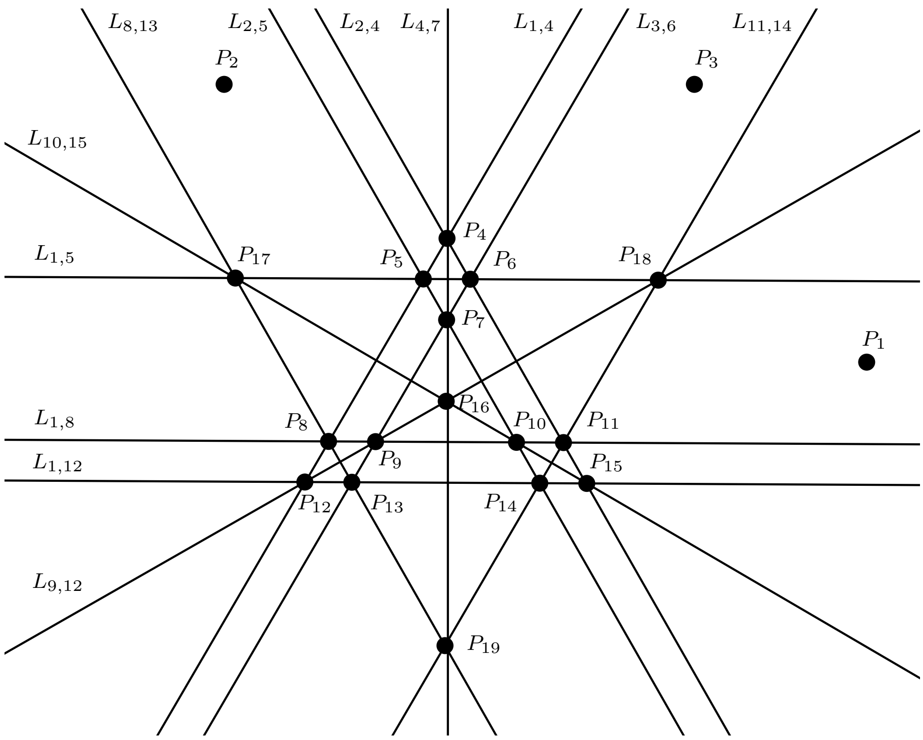

Figure 4.2: Arrangement C_7.

Here, using a projective automorphism, we may assume (with the notation as in the picture) that $P_1 = (1:0:0), P_2 = (-1:1:0), P_3 = (1:1:0)$ and $P_4 = (0:0:1)$. Then we have lines:

$$L_{2,4}: \; x + y = 0,$$
$$L_{1,4}: \; x - y = 0.$$

We need now to introduce the parameter to proceed with the construction, so take a point on the line $L_{1,4}$:

$$P_5 = (a : a : 1),$$

where $a \neq 0$. We get the lines

$$L_{1,5}: \; y - az = 0,$$
$$L_{2,5}: \; x + y - 2az = 0$$

and the point

$$P_6 = L_{2,4} \cap L_{1,5} = (-a : a : 1)$$

and then the line

$$L_{3,6}: \; x - y + 2az = 0.$$

To continue we need to choose next point. We take a point on the line $L_{2,5}$.

$$P_7 = (b : -b + 2a : 1).$$

We get the line

$$L_{4,7}: \; 2ax - bx - by = 0.$$

The condition for the lines $L_{4,7}, L_{2,5}, L_{3,6}$ to meet at P_7 is

$$ba = 0.$$

As $a \neq 0$, we have to take $b = 0$. Thus, from now on:

$$P_7 = (0 : 2a : 1)$$

and

$$L_{4,7}: \; 2ax = 0.$$

Again, we need a new parameter. Take a point on the line $L_{1,4}$

$$P_8 = (c : c : 1),$$

where $a \neq c, c \neq 0$. Then

$$L_{1,8}: \ y - cz = 0,$$

$$P_9 = L_{1,8} \cap L_{3,6} = (-2a + c : c : 1),$$
$$P_{10} = L_{1,8} \cap L_{2,5} = (2a - c : c : 1),$$
$$P_{11} = L_{1,8} \cap L_{2,4} = (-c : c : 1).$$

Now, choose the last parameter by taking a point, again on the line $L_{1,4}$:

$$P_{12} = (d : d : 1),$$

with d different from $0, a$ and c. Then

$$L_{1,12}: \ y - dz = 0,$$

$$P_{13} = L_{1,12} \cap L_{3,6} = (-2a + d : d : 1),$$
$$P_{14} = L_{1,12} \cap L_{2,5} = (2a - d : d : 1),$$
$$P_{15} = L_{1,12} \cap L_{2,4} = (-d : d : 1),$$
$$L_{10,15}: \ (c - d)x + (c - d - 2a)y + 2adz = 0,$$

$$P_{17} = L_{10,15} \cap L_{1,5} = (2a^2 - ac - ad : ac - ad : c - d),$$

$$L_{9,12}: \ (c - d)x + (2a - c + d)y - 2adz = 0,$$

$$P_{18} = L_{9,12} \cap L_{1,5} = (-2a^2 + ac + ad : ac - ad : c - d),$$

$$L_{8,13}: \ (c - d)x - (2a + c - d)y + 2acz = 0,$$
$$L_{11,14}: \ (c - d)x + (2a + c - d)y - 2acz = 0,$$

and finally

$$P_{19} = L_{8,13} \cap L_{11,14} = (0 : 4ac^2 - 4acd : 4ac - 4ad + 2c^2 - 4cd + 2d^2).$$

Almost all points of the construction are triple without any additional conditions. Only P_2 and P_3 require an additional condition to be triple, namely:

$$4a(a + c - d) = 0.$$

As $a \neq 0$, we get $a + c - d = 0$. Thus the parametrization space of this configuration is a line and the configuration has a realization over $\mathbb{Q}$. It is not difficult to check (with help of, e.g., Singular) that the product of all twelve lines (which obviously is in $I_7^{(3)}$) does not belong to I_7^2. Thus, the triple points of this configuration give another rational example of the non-containment of the third symbolic power into the second ordinary power of an ideal.
For the convenience of the reader, we enclose the Singular script below:

```
LIB "elim.lib";
ring R=(32003,a,d),(x,y,z),dp;
option(redSB);
proc rdideal(number p, number q, number r){
matrix m[2][3]=p,q,r,x,y,z;
ideal I=minor(m,2);
I=std(I);
return(I);}
proc pline(list P1, list P2){
matrix A[3][3]=P1[1],P1[2],P1[3],P2[1],P2[2],P2[3],x,y,z;
return(det(A));}
ideal P1=rdideal(1,0,0);
ideal P2=rdideal(-1,1,0);
ideal P3=rdideal(1,1,0);
ideal P4=rdideal(0,0,1);
ideal P5=rdideal(a,a,1);
ideal P6=rdideal(-a,a,1);
ideal P7=rdideal(0,2*a,1);
ideal P8=rdideal((d-a),(d-a),1);
ideal P9=rdideal(-2*a+(d-a),(d-a),1);
ideal P10=rdideal(2*a-(d-a),(d-a),1);
ideal P11=rdideal(-(d-a),(d-a),1);
ideal P12=rdideal(d,d,1);
ideal P13=rdideal(-2*a+d,d,1);
ideal P14=rdideal(2*a-d,d,1);
ideal P15=rdideal(-d,d,1);
ideal P16=rdideal(0,4*a*d,4*a-2*(d-a)+2*d);
ideal P17=rdideal(2*(a2)-a*(d-a)-a*d,a*(d-a)-a*d,(d-a)-d);
ideal P18=rdideal(-2*(a2)+a*(d-a)+a*d, a*(d-a)-a*d,(d-a)-d);
ideal P19=rdideal(0,4*a*((d-a)^2)-4*a*(d-a)*d,4*a*(d-a)-4*a*d+2*((d-a)^2)-4*(d-a)*d+2*(d2));
poly pp=(2*(d*z-y))*((d-a)*z-y)*(a*z-y)*(2*a*d*z-2*a*y+(d-a)*x+(d-a)*y-d*x-d*y)*
(2*a*(d-a)*z-2*a*y+(d-a)*x-(d-a)*y-d*x+d*y)*(2*a*z-x-y)*(-x-y)*x*a*(2*a*z+x-y)*
(-2*a*(d-a)*z+2*a*y+(d-a)*x+(d-a)*y-d*x-d*y)*(-2*a*d*z+2*a*y+(d-a)*x-(d-a)*y-d*x+d*y)*(-y+x);
ideal I=intersect(P1,P2,P3,P4,P5,P6,P7,P8,P9,P10,P11,P12,P13,P14,P15,P16,P17,P18,P19);
I=std(I);
reduce(pp,std(I^2));
```

Chapter 5

Freeness of special line arrangements

5.1 Freeness of Böröczky's line arrangements

We begin by showing that for some classes of line arrangements the weak combinatorics can be read of the Poincaré polynomial, see Definition 2.2.5. This is not true in general.

Proposition 5.1.1. *Let $\mathcal{A}, \mathcal{B}$ be two line arrangements having only double and triple points of intersection. Suppose that $\pi(\mathcal{A}, t) = \pi(\mathcal{B}, t)$, then $t_2(\mathcal{A}) = t_2(\mathcal{B})$ and $t_3(\mathcal{A}) = t_3(\mathcal{B})$.*

To begin with, let us emphasise here that we can obtain an analogous result replacing in the statement of Proposition 5.1.1 triple points by arbitrary k-fold points with a fixed $k \geq 3$.

Proof. Suppose that $\pi(\mathcal{A}, t) = \pi(\mathcal{B}, t)$. This implies in particular that

$$|\mathcal{A}| = |\mathcal{B}|$$

and

$$t_2(\mathcal{A}) + 2t_3(\mathcal{A}) = t_2(\mathcal{B}) + 2t_3(\mathcal{B}),$$

which gives

$$t_2(\mathcal{A}) - t_2(\mathcal{B}) = 2(t_3(\mathcal{B}) - t_3(\mathcal{A})).$$

Observe that

$$t_2(\mathcal{A}) + 3t_3(\mathcal{A}) = \binom{|\mathcal{A}|}{2} = \binom{|\mathcal{B}|}{2} = t_2(\mathcal{B}) + 3t_3(\mathcal{B}),$$

and this gives

$$t_2(\mathcal{A}) - t_2(\mathcal{B}) = 3(t_3(\mathcal{B}) - t_3(\mathcal{A})).$$

Since

$$3(t_3(\mathcal{B}) - t_3(\mathcal{A})) = t_2(\mathcal{A}) - t_2(\mathcal{B}) = 2(t_3(\mathcal{B}) - t_3(\mathcal{A})),$$

then this implies $t_3(\mathcal{A}) = t_3(\mathcal{B})$ and we also have $t_2(\mathcal{A}) = t_2(\mathcal{B})$, which completes the proof. □

Remark 5.1.2. In fact, we can obtain an analogous result replacing triple by arbitrary k-fold points with $k \geq 4$. The same argument works.

The main result of this section is the following classification result. Before we present our proof, let us recall Terao's factorization theorem [36] for free complex line arrangements.

Theorem 5.1.3 (Terao)**.** *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be a free line arrangement, then its Poincaré polynomial splits as*

$$\pi(\mathcal{L}; t) = (1 + t)(1 + d_1 t)(1 + d_2 t),$$

where d_i's are the exponents as in Definition 2.2.10.

Theorem 5.1.4. *Let $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$ be a line arrangement having only double and triple points as the intersections. Suppose that $\mathcal{A}$ is free, then $2 \leq |\mathcal{A}| \leq 9$.*

Proof. Suppose that $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$ is free and $|A| \geq 3$ as the case of two lines is obvious. Then by Terao's Theorem 5.1.3, the Poincaré polynomial $\pi(\mathcal{A}, t)$ splits into linear factors over the integers. Observe that

$$\pi(\mathcal{A}, t) = (1 + t) \cdot \Big(1 + (|\mathcal{A}| - 1)t + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}) + 1 - |\mathcal{A}|)t^2\Big),$$

and the quadratic factor also splits into linear factors. This condition implies that

$$(|\mathcal{A}| - 1)^2 - 4t_2(\mathcal{A}) - 8t_3(\mathcal{A}) + 4|\mathcal{A}| - 4 \geq 0.$$

Using the combinatorial equality

$$|\mathcal{A}|(|\mathcal{A}| - 1) = 2t_2(\mathcal{A}) + 6t_3(\mathcal{A})$$

one gets

$$3|\mathcal{A}| - 3 \geq 2t_2(\mathcal{A}) + 2t_3(\mathcal{A}).$$

and

$$|\mathcal{A}|(|\mathcal{A}| - 1) = 2t_2(\mathcal{A}) + 6t_3(\mathcal{A}) \leq 3(2t_2(\mathcal{A}) + 2t_3(\mathcal{A})) \leq 9|\mathcal{A}| - 9.$$

This provides the condition $|\mathcal{A}| \leq 9$, and finally we obtain $3 \leq |\mathcal{A}| \leq 9$, which completes the proof. □

Corollary 5.1.5. *Except the cases $n = 4, 5, 6$, Böröczky arrangements of n lines are not free.*

Proof. By the above result, it is enough to check the cases $n \in \{4, 5, 6, 7, 8, 9\}$. Since

$$\pi(\mathcal{A}, t) = 1 + |\mathcal{A}|t + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}))t^2 + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}) + 1 - |\mathcal{A}|)t^3,$$

we obtain the following table

$n = \|\mathcal{A}\|$	$t_2(\mathcal{A})$	$t_3(\mathcal{A})$	$\pi(\mathcal{A}, t)$
4	3	1	$1 + 4t + 5t^2 + 2t^3 = (t + 1)^2(2t + 1)$
5	4	2	$1 + 5t + 8t^2 + 4t^3 = (t + 1)(2t + 1)^2$
6	3	4	$1 + 6t + 11t^2 + 6t^3 = (t + 1)(2t + 1)(3t + 1)$
7	6	5	$1 + 7t + 16t^2 + 10t^3 = (t + 1)(10t^2 + 6t + 1)$
8	6	5	$1 + 8t + 21t^2 + 14t^3 = (t + 1)(14t^2 + 7t + 1)$
9	6	10	$1 + 9t + 26t^2 + 18t^3 = (t + 1)(18t^2 + 8t + 1)$

It is easy to see that for $n \in \{7, 8, 9\}$ we cannot factorize Poincaré polynomials into linear factors over the integers, so in these cases Böröczky arrangements are not free. When $n = 6$, then this arrangement is projectively equivalent to the well-known arrangement $\mathcal{A}_1(6)$, which is known to be free. Now we focus on the remaining cases $n \in \{4, 5\}$ showing for them freeness explicitly. We are going to use Definition 2.2.10, if $\mathcal{L}$ is free, then we have the following resolution

$$0 \to \oplus_{i=1,2} S(-d_i - (n - 1)) \to S^3(-n + 1) \to S.$$

If $n = 4$, then the defining equation of $\mathbb{B}_4$ has the form

$$Q_4(x, y, z) = xy(y - x + z)(y + x - z).$$

Denote by $J_{\mathbb{B}_4}$ the Jacobian ideal generated by the partials of Q_4, and by $S/J_{\mathbb{B}_4}$ the Milnor algebra. Then the resolution of $S/J_{\mathbb{B}_4}$ has the following form:

$$0 \to S(-4) \oplus S(-5) \to S^3(-3) \to S,$$

with the following relations:

$$3x \cdot \partial_x Q_4 - y \cdot \partial_y Q_4 + (4x - z) \cdot \partial_z Q_4 = 0,$$

$$(4x^2 - 7xz) \cdot \partial_x Q_4 + (13yz - 12xy) \cdot \partial_y Q_4 + (16y^2 - 3z^2) \cdot \partial_z Q_4 = 0.$$

This allows us to conclude that $\mathbb{B}_4$ is free.

Now consider the case $n = 5$. As the coordinates of $P_0, ..., P_9$ (vertices of a regular 10-gon) in this case satisfy the condition $\{P_0, ..., P_9\} \subseteq \{(\pm(\frac{1}{4}\sqrt{5} \pm \frac{1}{4}), \pm\frac{1}{4}\sqrt{10 \pm 2\sqrt{5}})\}$, it is more convenient to change the coordinates and consider the following equivalent arrangement of lines given by

$$Q_5(x, y, z) = y(2y - x)(2y + 3x)(x + 2y - 4z)(3x - 2y - 12z).$$

The minimal resolution of $S/J_{\mathbb{B}_5}$ has the following form:

$$0 \to S(-6) \oplus S(-6) \to S^3(-4) \to S,$$

with the following syzygies:

$$(3x^2 - 20xy - 20y^2 + 24xz) \cdot \partial_x Q_5 + (24yz - 12xy) \cdot \partial_y Q_5 + (18xz + 20yz - 10xy - 36z^2) \cdot \partial_z Q_5 = 0,$$

$$(176xy + 160y^2 - 288xz) \cdot \partial_x Q_5 + (120xy + 16y^2 - 288yz) \cdot \partial_y Q_5$$
$$+(15x^2 + 100xy - 20y^2 - 240xz - 224yz + 432z^2) \cdot \partial_z Q_5 = 0,$$

which tells us that $\mathbb{B}_5$ is also free. □

At the end, it is worth pointing out that Dimca and Sernesi in [12] were able to obtain a similar result about line arrangements with double and triple points using much deeper methods. However, our approach is completely different and much simpler using only combinatorial methods related to basics on line arrangements.

5.2 Extensions to the supersolvability

In the last section, we focus on a natural problem that one can consider in the case of Böröczky's arrangements. As we observed previously, near-pencils in the complex projective plane are free arrangements since these examples are supersolvable. Using this idea, let $\mathcal{A}$ be an arrangement of lines (which is not a pencil or near-pencil, in order to avoid trivialities). Then we add one vertex and we join it with all singular points $\mathrm{Sing}(\mathcal{A})$ by lines. In result we obtain a new arrangement $\mathcal{A}^{CS}$ which is by the construction supersolvable. This is a trivial extension, but we can do much better, i.e., start with a vertex V chosen from $\mathrm{Sing}(\mathcal{A})$ we add some lines through V in order to force our new arrangement to be supersolvable. This motivates the introducing the following constant:

$$\mathrm{extSS}(\mathcal{A}) = \min\{d : d \text{ is the number of lines that we add to } \mathcal{A} \text{ so that } \mathcal{A}^{CS} \text{ is supersolvable}\}.$$

Remark 5.2.1. Note that the number $\mathrm{extSS}(\mathcal{A})$ is well-defined and finite.

Proof. Let $\mathcal{A}$ be an arbitrary line arrangement and P a general point in the plane. Let $\mathcal{B}$ be the arrangement consisting of all lines in $\mathcal{A}$ and lines joining P with all intersection points of $\mathcal{A}$. Then $\mathcal{B}$ is supersolvable. □

It follows immediately that for any arrangement $\mathcal{A}$ one has

$$\mathrm{extSS}(\mathcal{A}) \leq |\mathrm{Sing}(\mathcal{A})|.$$

Note that this upper bound is very rough. For example, if $\mathcal{A}$ is already supersolvable, then $\mathrm{extSS}(\mathcal{A}) = 0$. It is thus natural to state the following question.

Problem 5.2.2. Find $\mathrm{extSS}(\cdot)$ numbers for relevant line arrangements.

By the way of warming up, we study star configurations of lines.

Example 5.2.3. Let $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of d generic lines. Then

$$\mathrm{extSS}(\mathcal{A}) = \binom{d-2}{2}.$$

Indeed, let P be any double intersection point of $\mathcal{A}$ and let ℓ, m be the lines from $\mathcal{A}$ intersecting at P. We denote by $\mathcal{A}' = \mathcal{A} \setminus \{\ell, m\}$. Then we need to join singular points of $\mathcal{A}'$ with P. There are exactly $\binom{d-2}{2}$ such points. Note that there are no additional collinearities because $\mathcal{A}$ is a generic arrangement.

Before we proceed to the main results of this section, let us present also another motivation that leads us to study the mentioned extensions to the supersolvability property. Very recently, the theory of unexpected curves has appeared and gained a lot of attention by researchers.

Let $\mathcal{P} = \{P_1, ..., P_s\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a finite set of points and denote by $\bar{m} = \{m_1, ..., m_s\}$ a sequence of positive integers called the sequence of multiplicities. Denote by $X = m_1P_1 + ... + m_sP_s$ a fat point scheme and consider the associated ideal

$$I(X) = \bigcap_{i=1}^{s} I(P_i)^{m_i}.$$

Here by $I(X)_d$ we mean the homogeneous component of degree d. Now we can define the expected dimension by

$$\operatorname{expdim} I(X)_d = \max\left\{\binom{d+2}{2} - \sum_{i=1}^{s}\binom{m_i+1}{2}, 0\right\}.$$

Geometrically speaking, the vector space $I(X)_d$ is the linear system of plane curves of degree d passing through each point P_i's with multiplicity at least m_i. The expected dimension informs us whether we can expect the existence of such curves, and in general $\dim I(X)_d \geq \operatorname{expdim} I(X)_d$.

Definition 5.2.4. Let d be a non-negative integer. We say that a finite set of points Z in the complex projective plane admits an unexpected curve in degree d with a general point P of multiplicity $d-1$ if

$$\dim(I(Z+(d-1)P))_d > \max\left\{\dim I(Z)_d - \binom{d}{2}, 0\right\}.$$

Definition 5.2.5. We say that an arrangement of lines $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ admits an unexpected curve if the set of points Z is the dual configuration to lines in $\mathcal{L}$.

In the context of supersolvable line arrangements, Di Marca, Malara, and Oneto [10] proved the following result.

Theorem 5.2.6. *A supersolvable line arrangement $\mathcal{L}$ admits an unexpected curve if and only if $d > 2m$ where d is the number of lines and m is the maximum multiplicity of an intersection point of the lines in $\mathcal{L}$.*

This theorem provides as a very nice criterion for the existence of unexpected curves and once we are able to extend a well-known arrangement in such a way that the resulting object is supersolvable and satisfies the condition that $d > 2m$, then we have a new example of an unexpected curve. Let us start with a baby-case of Fermat arrangements of lines.

Example 5.2.7. Fermat arrangement of lines is defined in the complex projective plane by the linear factors of the following polynomial

$$F(x,y,z) = (x^n - y^n)(y^n - z^n)(z^n - x^n),$$

where $n \geq 3$. It is well-known that the arrangement consists of $3n$ lines and $t_n = 3$, $t_3 = n^2$. It is easy to observe that this is not a supersolvable arrangement since the three fundamental points (which are the intersection points) cannot be joined by lines from the arrangement. One of the smallest extension of Fermat arrangements looks as follows:

$$\widetilde{F}(x,y,z) = xy(x^n - y^n)(y^n - z^n)(z^n - x^n),$$

where as previously $n \geq 3$. The new arrangement consists of $3n+2$ lines and

$$t_2 = 2n, \quad t_3 = n^2, \quad t_{n+1} = 2, \quad t_{n+2} = 1.$$

This arrangement is clearly supersolvable and the exponents are $d_1 = n+1$, $d_2 = 2n$. Moreover, we see that $3n+2 = d > 2m = 2n+4$, since $n \geq 3$, so our new family of line arrangements, denoted in the literature by $\mathcal{A}_3^2(n)$, leads to new examples of unexpected curves.

Let us come back to the numbers extSS$(\cdot)$. Using some particular symmetries of Böröczky arrangements of $n = 6k$ lines with $k \geq 2$ we can show the following result. Observe in the meantime that for $k = 1$ our arrangement $\mathcal{B}_6$ is obviously supersolvable.

Theorem 5.2.8. *Let $n = 6k$ for $k \geq 2$. Then*

$$\text{extSS}(\mathcal{B}_{6k}) \leq 6k^2 - 6k.$$

In our construction, the arrangements $\mathcal{B}_{6k}^{CS}$ have $6k^2$ lines and the following weak combinatorics:

$$t_{3+6k^2-6k} = 1, \quad t_4 = 6(k-1)^2, \quad t_3 = 15k - 12, \quad t_2 = 36k^3 - 72k^2 + 42k - 3.$$

Finally, the exponents of free arrangement $\mathcal{B}_{6k}^{CS}$ are $d_1 = 6k-3$, $d_2 = 6k^2 - 6k + 2$.

Proof. Here we present a detailed sketch of our construction. Take one of the three headlights = points of multiplicity 3 of $\mathcal{B}_{6k}$ and denote this point by O. Take the three lines passing through O. Observe that each of the three lines possesses exactly $3k$ singular points from the arrangement. Since the only intersection point of the three lines is O, then on the three lines we have altogether exactly $9k-2$ intersection points, among them exactly 3 double points. Now we construct our extension $\mathcal{B}_{6k}^{CS}$ by joining O with each singular point except those $9k-2$ lying on the three lines. A simple calculation tells us that we add the following number of lines

$$6k - 3 + \frac{6k(6k-3)}{6} + 1 - (9k-2) = 6k - 3 + 6k^2 - 3k + 1 - 9k + 2 = 6k^2 - 6k.$$

Now we can describe the combinatorics of $\mathcal{B}_{6k}^{CS}$. By the construction, the vertex O has multiplicity $6k^2-6k+3$, and this is the only point of such multiplicity. Next, we obtain quadruple points by joining the previous triple points with the vertex O, we have altogether

$$t_4 = 6k^2-3k+1-(9k-5) = 6k^2-12k+6 = 6(k-1)^2.$$

We get also new triple points (out of old double points), there are exactly $6k-6$ such points. Altogether we have

$$t_3 = 6k-6+9k-5-1 = 15k-12,$$

where the last -1 in the middle equality comes from the fact that O is no longer a triple point. Finally, we can compute the number of double points. Using the combinatorial count we obtain that

$$t_2 = \frac{6k^2(6k^2-1)}{2} - 3\cdot(15k-12) - 6\cdot(6(k-1)^2) - \binom{6k^2-6k+3}{2} = 36k^3-72k^2+42k-3.$$

By the construction, $\mathcal{B}_6^{CS}$ is supersolvable and by Jambu-Terao's result [24], the arrangement is free. We compute the exponents of the arrangement with use of the Poincaré polynomial. Observe that

$$\pi(B_6^{CS};t) = (1+t)\Big(1+(6k^2-1)t+(36k^3-54k^2+30k-6)t^2\Big).$$

Since $\triangle_t = (6k^2-12k+5)^2$ and $6k^2-12k+5$ is non-negative for $k \geq 2$, we can compute rational roots of the polynomial, namely

$$a_1 = \frac{-6k^2+1+6k^2-12k+5}{12(2k-1)(3k^2-3k-1)} = \frac{-(2k-1)}{2(2k-1)(3k^2-3k+1)} = \frac{-1}{6k^2-6k+2},$$

$$a_2 = \frac{-6k^2+1-6k^2+12k-5}{12(2k-1)(3k^2-3k-1)} = \frac{-12k^2+12k-4}{12(2k-1)(3k^2-3k+1)} = \frac{-1}{3(2k-1)}.$$

This gives us finally that

$$\pi(\mathcal{B}_6^{CS};t) = (1+t)(1+(6k-3)t)(1+(6k^2-6k+2)t),$$

and the exponents are $d_1 = 6k-3$, $d_2 = 6k^2-6k+2$. □

Now we turn to the Klein arrangement of lines $\mathcal{K}$ (see [26]). We determine explicitly an upper bound on the value of $\mathrm{extSS}(\mathcal{K})$. Let us recall that the arrangement $\mathcal{K}$ consists of $d=21$ lines and $t_3=28$, $t_4=21$.

Proposition 5.2.9. *For the Klein arrangement of lines* $\mathcal{K}$ *we have* $\mathrm{extSS}(\mathcal{K}) \leq 20$.

Proof. Let us recall the most crucial fact about Klein's arrangement of lines about the singular points. It can be observed that each line from the arrangement contains exactly 4 triple and 4 quadruple singular points. Choose one of the quadruple points (due to the mentioned remarkable property) and the four lines passing through it. Denoting this quadruple point by O, we observe that these four lines contain exactly $4 \cdot 8 - 3 = 29$ singular points, so we are left with 12 triple points and 8 quadruple points. Next, we join each of the remaining 20 singular points with O, so altogether our line arrangement $\mathcal{K}^{CS}$ consists of $21 + 20 = 41$ lines and it has the following weak combinatorics:

$$t_{24} = 1, \quad t_5 = 8, \quad t_4 = 24, \quad t_3 = 16, \quad t_2 = 272.$$

By the construction, $\mathcal{K}^{CS}$ is supersolvable, and we can compute the exponents. Observe that

$$\pi(\mathcal{K}^{CS}; t) = (1+t)\left(1 + 40t + 391t^2\right) = (1+t)(1+17t)(1+23t),$$

so the exponents are $d_1 = 17$, $d_2 = 23$.

□

Finally, we consider the last arrangement of our interests, namely the Wiman arrangement of lines [37], denoted by $\mathcal{W}$. This remarkable arrangement consists of 45 lines and it has

$$t_3 = 120, \quad t_4 = 45, \quad t_5 = 36.$$

Proposition 5.2.10. *For the Wiman arrangement of lines* $\mathcal{W}$ *one has* $\mathrm{extSS}(\mathcal{W}) \leq 125$.

Proof. Let us recall the most crucial fact about the singular points of Wiman's arrangement of lines. We observed that each line from the arrangement contains exactly 4 quintuple, 4 quadruple, and 8 triple singular points. Choose one of the quintuple points and the five lines passing through this point. Denoting this point by O, we observe that these five lines contain exactly 76 singular points, so we are left with 80 triple points, 25 quadruple points, and 20 quintuple points. Next, we join each of the mentioned singular points with O, so altogether our line arrangement $\mathcal{W}^{CS}$ consists of $45 + 125 = 170$ lines and it has the following weak combinatorics:

$$t_{130} = 1, \quad t_6 = 20, \quad t_5 = 40, \quad t_4 = 100, \quad t_3 = 40, \quad t_2 = 4560.$$

By the construction, $\mathcal{W}^{CS}$ is supersolvable, and we can compute the exponents. Observe that

$$\pi(\mathcal{W}^{CS};t) = (1+t)\Big(1 + 169t + 5160t^2\Big) = (1+t)(1+40t)(1+129t),$$

so the exponents are $d_1 = 40$, $d_2 = 129$. $\square$

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