AKADEMIA PEDAGOGICZNA im. KOMISJI EDUKACJI NARODOWEJ

63

Annales Academiae Paedagogicae Cracoviensis

Studia Mathematica VII

63

Annales Academiae Paedagogicae Cracoviensis

Studia Mathematica VII

Wydawnictwo Naukowe
Akademii Pedagogicznej
Kraków 2008

Contact: studmath@up.krakow.pl
Available online at http://studmath.up.krakow.pl

Contents

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Bożena Piątek

Riemann integrability of a nowhere continuous multifunction

Abstract. We present an example of the Riemann integrable multifunction which is discontinuous at each point with respect to the Hausdorff metric. The constructed multifunction is neither lower nor upper semicontinuous.

1. Introduction

The Riemann integral for multifunctions with compact convex values was investigated by A. Dinghas [3] and M. Hukuhara [4]. Some properties of Riemann integral of multifunctions with convex closed bounded values may be found in [5]. The Riemann integrability of multifunctions with compact convex values was presented in [6].

Our main goal is to show that the continuity for almost all $x \in [a,b]$ of a bounded multifunction is not a necessary condition for the Riemann integrability. The same example shows also that the monotonicity does not imply the almost everywhere continuity of multifunctions.

2. Basic definitions

Let X be a real Banach space. Denote by $clb(X)$ the set of all nonempty convex closed bounded subsets of X. For given $A, B \in clb(X)$, we set

$$A + B = \{a + b \mid\ a \in A,\ b \in B\},$$
$$\lambda A = \{\lambda a \mid\ a \in A\} \qquad \text{for } \lambda \geq 0$$

and

$$A \overset{*}{+} B = \operatorname{cl}(A + B),$$

where $\operatorname{cl} A$ means the closure of A in X. It is easy to see that $(clb(X), \overset{*}{+}, \cdot)$ satisfies the following properties

AMS (2000) Subject Classification: 28B20, 26A48.

$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B,$$
$$(\lambda + \mu)A = \lambda A \overset{*}{+} \mu A,$$
$$\lambda(\mu A) = (\lambda\mu)A,$$
$$1 \cdot A = A$$

for each $A, B \in clb(X)$ and $\lambda, \mu \geq 0$. If $A, B, C \in clb(X)$, then the equality $A \overset{*}{+} C = B \overset{*}{+} C$ implies $A = B$, thus $clb(X)$ with addition $\overset{*}{+}$ satisfies the cancellation law (see [1, Theorem II-17] and [8, Corollary 2.3]).

$clb(X)$ is a metric space with the Hausdorff metric h defined by the relation

$$h(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} \|a - b\|$. The metric space $(clb(X), h)$ is complete (see e.g. [1, Theorem II-3]). Moreover, the Hausdorff metric h is translation invariant since

$$h(A \overset{*}{+} C, B \overset{*}{+} C) = h(A + C, B + C) = h(A, B)$$

(cf. [7, Lemma 3], [2, Lemma 2.2]) and positively homogeneous

$$h(\lambda A, \lambda B) = \lambda h(A, B)$$

for all $A, B, C \in clb(X)$ and $\lambda \geq 0$ (cf. [2, Lemma 2.2]).

Lemma 1

Let X be a normed vector space. If $A, B, C \in clb(X)$ and $A \subset B \subset C$, then

$$h(B, C) \leq h(A, C) \qquad \textit{and} \qquad h(A, B) \leq h(A, C).$$

Proof. Since $e(B, C) = 0$ and $d(c, B) \leq d(c, A)$, $c \in C$, we have

$$h(B, C) = e(C, B) \leq e(C, A) = h(A, C).$$

The proof of the second inequality is analogous.

Let F be a multifunction defined on the interval $[a, b]$ with nonempty convex closed bounded values in X. A set $\Delta = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \ldots < x_n = b$, is said to be a *partition* of $[a, b]$. For a given partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ we set $\delta(\Delta) = \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}$. Δ' is said to be a *subpartition* of Δ if Δ' is a partition of the same interval and $\Delta \subset \Delta'$. For the partition Δ and for a system $\tau = (\tau_1, \ldots, \tau_n)$ of intermediate points $\tau_i \in [x_{i-1}, x_i]$ we create the *Riemann sum*

$$S(\Delta, \tau) = (x_1 - x_0)F(\tau_1) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(\tau_n).$$

If for every sequence $((\Delta^\nu, \tau^\nu))$, where $\Delta^\nu = \{x_0^\nu, x_1^\nu, \dots, x_{n_\nu}^\nu\}$ are partitions of $[a,b]$ such that $\lim_{\nu\to\infty} \delta(\Delta^\nu) = 0$ and $\tau^\nu = (\tau_1^\nu, \dots, \tau_{n_\nu}^\nu)$ are systems of intermediate points ($\tau_i^\nu \in [x_{i-1}^\nu, x_i^\nu]$), the sequence of the Riemann sums $(S(\Delta^\nu, \tau^\nu))$ tends to the same limit I with respect to the Hausdorff metric, then F is said to be *Riemann integrable* over $[a,b]$ and $I =: \int_a^b F(x)\,dx$. Obviously, if the limit I exists, then $I \in clb(X)$.

LEMMA 2
Let X be a real Banach space and $F\colon [a,b] \longrightarrow clb(X)$. Then the following conditions are equivalent:

(i) *F is Riemann integrable on $[a,b]$;*

(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition Δ satisfying $\delta(\Delta) < \delta$, for every subpartition Δ' of Δ and for all corresponding systems τ, τ' of intermediate points, the inequality*

$$h\left(S(\Delta,\tau), S(\Delta',\tau')\right) < \varepsilon$$

is satisfied.

The easy proof is omitted. The completeness of $(clb(X), h)$ is needed only in the proof of sufficiency.

We say that a multifunction $F\colon [a,b] \longrightarrow clb(X)$ is *increasing* if

$$F(s) \subset F(t)$$

holds true for all $a \le s \le t \le b$.

PROPOSITION 1
An increasing multifunction $F\colon [a,b] \longrightarrow clb(X)$ is right-hand side lower semi-continuous at each point of the interval $[a,b)$.

Proof. Let $t_0 \in [a,b)$ and let U be an open subset of X such that $F(t_0) \cap U \neq \emptyset$. Since $F(t_0) \subset F(t)$ when $t > t_0$, $F(t) \cap U \neq \emptyset$ for each $t \in [t_0, b]$ which implies the right-hand side lower semi-continuity of F at t_0.

PROPOSITION 2
An increasing multifunction $F\colon [a,b] \longrightarrow clb(X)$ is left-hand side upper semi-continuous at each point of the interval $(a,b]$.

Proof. Let $t_0 \in (a,b]$ and let U be an open subset of X such that $F(t_0) \subset U$. Since $F(t) \subset F(t_0)$ for $t \in [a, t_0]$, $F(t) \subset U$ for the same t and F is left-hand side upper semi-continuous at t_0.

For an increasing multifunction $F\colon [a,b] \longrightarrow clb(X)$ and for each partition $\Delta = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ we may create two sums

$$S(\Delta) := (x_1 - x_0)F(x_1) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(x_n)$$

and

$$s(\Delta) := (x_1 - x_0)F(x_0) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(x_{n-1}).$$

3. Main results

Let Y be a Banach space defined as the set of all bounded functions $x\colon [0,1] \longrightarrow \mathbb{R}$ with the norm $||x|| = \sup_{t\in[0,1]} |x(t)|$.

Let $F\colon [0,1] \longrightarrow 2^Y$ be a multifunction with values defined as follows

$$F(t) := \{x\colon [0,1] \longrightarrow [0,1] \mid x(s) = 0 \text{ for all } s > t\}, \qquad t \in [0,1),$$
$$F(1) := \{x \in Y \mid x\colon [0,1] \longrightarrow [0,1]\}.$$

In particular, $F(0)$ is equal to $\{x\colon [0,1] \longrightarrow [0,1] \mid x(s) = 0 \text{ for each } s \in (0,1]\}$. It is not difficult to see that the set $F(t)$ is an element of $clb(Y)$ for all $t \in [0,1]$.

Now we consider some properties of the multifunction F.

Remark 1
F is increasing on $[0,1]$. Indeed, let $s < t$ and $s,t \in [0,1]$. If $t = 1$, then $F(s) \subset F(1)$ for all $s \in [0,1]$. Assume that $t < 1$ and $x \in F(s)$. We have $x(u) = 0$ for all $u > s$ and, in particular, for all $u > t$. Consequently $F(s) \subset F(t)$.

Remark 2
By Proposition 1 and Remark 1 the multifunction F is right-hand side lower semi-continuous at each point of $[0,1)$. We shall show that it is not left-hand side lower semi-continuous in $(0,1]$. Let $t_0 \in (0,1]$. Define $x(t) = 1$ for $t \in [0,t_0]$ and $x(t) = 0$ for $t \in (t_0,1]$, if $t_0 \in (0,1)$. Let $S(x,\frac{1}{2})$ be an open ball in Y centered at x with the radius $\frac{1}{2}$. Of course $S(x,\frac{1}{2}) \cap F(t_0) \neq \emptyset$. Now take an arbitrary $s \in [0,t_0)$. If $y \in F(s)$, then

$$1 \geq \|x - y\| = \sup_{u\in[0,1]} |x(u) - y(u)| \geq \sup_{u\in(s,t_0)} |x(u) - y(u)| = 1.$$

Consequently, $\|x - y\| = 1$ and $y \notin S(x,\frac{1}{2})$, i.e., $S(x,\frac{1}{2}) \cap F(s) = \emptyset$ for all $0 \leq s < t_0$.

Remark 3
By Proposition 2 and Remark 1 the multifunction F is left-hand side upper semi-continuous at each point of the interval $(0,1]$. We will show that F is not right-hand side upper semi-continuous in $[0,1)$. Indeed, let $t_0 \in [0,1)$ and let U be an open set defined by $U = \bigcup_{x\in F(t_0)}\{y \in Y \mid \ \|y - x\| < \frac{1}{2}\}$. It is clear that $F(t_0) \subset U$, but $F(t) \not\subset U$ for each $t > t_0$. It is sufficient to choose $z \in F(t)$ such that $z(u) = 1$ for $u \in [t_0,t]$. Thus for each $x \in F(t_0)$

$$\|z - x\| \geq \sup_{u \in [t_0, t]} |z(u) - x(u)| = 1$$

and consequently $z \notin U$.

Remark 4
F is non-continuous at each point of the interval $[0, 1]$ with respect to the Hausdorff metric.

By Remark 2 it follows that $h(F(s), F(t)) = 1$ for all $s, t \in [0, 1]$ such that $0 \leq s < t \leq 1$. Thus $h(F(s), F(t)) = 1$ if $s \neq t$ and $\lim_{t \to s} h(F(s), F(t)) = 1$ for all $s \in [0, 1]$.

Theorem 1
The multifunction F defined by formulas

$$F(t) := \{x\colon [0,1] \longrightarrow [0,1] \mid \ x(s) = 0 \textit{ for all } s > t\}, \qquad t \in [0, 1)$$

and

$$F(1) := \{x \in Y \mid \ x\colon [0,1] \longrightarrow [0,1]\}$$

is Riemann integrable on $[0, 1]$.

Proof. Let $\varepsilon > 0$ and let $\Delta = \{t_0, t_1, \ldots, t_n\}$ be an arbitrary partition of $[0, 1]$ such that $\delta(\Delta) < \varepsilon$. It is sufficient (see Lemma 2) to show that for each subpartition Δ' and for each systems of intermediate points τ, τ' corresponding to Δ, Δ', respectively,

$$h\left(S(\Delta, \tau), S(\Delta', \tau')\right) < 2\varepsilon.$$

At first we are going to show that

$$\begin{aligned} s(\Delta) = \{x\colon [0,1] \longrightarrow [0,1] \mid \ x(t) \in [0, 1 - t_k] \text{ for } t \in (t_{k-1}, t_k],& \\ k \in \{1, \ldots, n\} \text{ and } x(0) \in [0,1]\},& \end{aligned} \tag{1}$$

$$\begin{aligned} S(\Delta) = \{y\colon [0,1] \longrightarrow [0,1] \mid y(t) \in [0, 1 - t_{k-1}] \text{ for } t \in (t_{k-1}, t_k],& \\ k \in \{1, \ldots, n\} \text{ and } y(0) \in [0,1]\}.& \end{aligned} \tag{2}$$

Let us take $a \in s(\Delta)$. We can find n sequences (a_k^ν), such that $a_k^\nu \in (t_k - t_{k-1})F(t_{k-1})$, and $\sum_{k=1}^n a_k^\nu \to a$ if $\nu \to \infty$. Obviously $a_k^\nu(t) \in [0, t_k - t_{k-1}]$ for $t \leq t_{k-1}$ and $a_k^\nu(t) = 0$ if $t > t_{k-1}$. Summing up over $k \in \{1, \ldots, n\}$ we have

$$0 \leq \left(\sum_{k=1}^n a_k^\nu\right)(t) = \sum_{k=1}^n a_k^\nu(t) = \sum_{j=k+1}^n a_j^\nu(t) \leq \sum_{j=k+1}^n (t_j - t_{j-1}) = 1 - t_k$$

for each $t \in (t_{k-1}, t_k]$ and

$$0 \leq \Big(\sum_{k=1}^{n} a_k^{\nu} \Big)(0) = \sum_{k=1}^{n} a_k^{\nu}(0) \leq \sum_{k=1}^{n} (t_k - t_{k-1}) = 1.$$

Thus $\sum_{k=1}^{n} a_k^{\nu}$ belong to the right-hand side of (1) which is a closed set, so a also belongs there.

Conversely, let a belongs to the right-hand side of (1). We define functions $b\colon [0,1] \longrightarrow [0,1]$, $b_k\colon [0,1] \longrightarrow [0,1]$, $k \in \{0,\dots,n-1\}$, by formulas

$$b(t) = \begin{cases} a(t), & t = 0, \\ \dfrac{a(t)}{1-t_k}, & t \in (t_{k-1}, t_k],\ k = 1,\dots,n-1, \\ 0, & t \in (t_{n-1}, 1] \end{cases}$$

and

$$b_k(t) = \begin{cases} b(t), & t \in [0, t_k], \\ 0, & t \in (t_k, 1]. \end{cases}$$

Obviously, $b_k \in F(t_k)$ for each $k \in \{0,\dots,n-1\}$.

For $t \in (t_{j-1}, t_j]$, where $j \in \{1,\dots,n-1\}$, we have

$$\begin{aligned} &[(t_1 - t_0)b_0 + \ldots + (t_n - t_{n-1})b_{n-1}](t) \\ &\quad = [(t_{j+1} - t_j)b_j + \ldots + (t_n - t_{n-1})b_{n-1}](t) \\ &\quad = [(t_{j+1} - t_j)b + \ldots + (t_n - t_{n-1})b](t) \\ &\quad = (1 - t_j)b(t) \\ &\quad = a(t), \end{aligned}$$

and for $u \in (t_{n-1}, t_n]$ the equality

$$[(t_1 - t_0)b_0 + \ldots + (t_n - t_{n-1})b_{n-1}](u) = 0 = a(u)$$

holds. Moreover

$$[(t_1 - t_0)b_0 + \ldots + (t_n - t_{n-1})b_{n-1}](0) = b(0) = a(0).$$

Thus $a \in s(\Delta)$ and the proof of (1) is complete.

The equality (2) can be established similarly.

Since F is increasing (by Remark 1) the following inclusions are valid

$$s(\Delta) \subset S(\Delta, \tau) \subset S(\Delta). \tag{3}$$

We will show that

$$s(\Delta) \subset S(\Delta', \tau') \subset S(\Delta). \tag{4}$$

There is no loss of generality in assuming that $\Delta' = \Delta \cup \{u\}$, where $u \in (t_{n-1}, 1)$ and $\tau' = (\tau_1, \ldots, \tau_{n+1})$, where $\tau_i \in [t_{i-1}, t_i]$, $i \in \{1, \ldots, n-1\}$, $\tau_n \in [t_{n-1}, u]$, $\tau_{n+1} \in [u, 1]$. By definitions of $s(\Delta)$, $S(\Delta)$ and $S(\Delta', \tau')$ we have

$$\begin{aligned} s(\Delta) &= (t_1 - t_0)F(t_0) \overset{*}{+} \ldots \overset{*}{+} (t_{n-1} - t_{n-2})F(t_{n-2}) \\ &\quad \overset{*}{+} (u - t_{n-1})F(t_{n-1}) \overset{*}{+} (t_n - u)F(t_{n-1}) \\ &\subset (t_1 - t_0)F(\tau_1) \overset{*}{+} \ldots \overset{*}{+} (t_{n-1} - t_{n-2})F(\tau_{n-1}) \\ &\quad \overset{*}{+} (u - t_{n-1})F(\tau_n) \overset{*}{+} (t_n - u)F(\tau_{n+1}) \\ &= S(\Delta', \tau') \\ &\subset (t_1 - t_0)F(t_1) \overset{*}{+} \ldots \overset{*}{+} (t_{n-1} - t_{n-2})F(t_{n-1}) \\ &\quad \overset{*}{+} (u - t_{n-1})F(t_n) \overset{*}{+} (t_n - u)F(t_n) \\ &= S(\Delta). \end{aligned}$$

Now, by Lemma 1, (3) and (4) we have

$$\begin{aligned} h\left(S(\Delta, \tau), S(\Delta', \tau')\right) &\le h\left(S(\Delta, \tau), S(\Delta)\right) + h\left(S(\Delta), S(\Delta', \tau')\right) \\ &\le 2h\left(S(\Delta), s(\Delta)\right). \end{aligned}$$

We are going to show that

$$e\left(S(\Delta), s(\Delta)\right) = \delta(\Delta).$$

Let $x_0, y_0 \colon [0,1] \longrightarrow [0,1]$ be defined by

$$x_0(t) = \begin{cases} 1, & t = 0, \\ 1 - t_k\,, & t \in (t_{k-1}, t_k], \end{cases} \qquad y_0(t) = \begin{cases} 1, & t = 0, \\ 1 - t_{k-1}, & t \in (t_{k-1}, t_k]. \end{cases}$$

Obviously $x_0 \in s(\Delta)$ and $y_0 \in S(\Delta)$.

In order to see that

$$\|y_0 - x_0\| = d\left(y_0, s(\Delta)\right) \tag{5}$$

suppose that $x \in s(\Delta)$. Then for $t = 0$ we have

$$y_0(t) - x(t) = 1 - x(t) \ge 0 = y_0(t) - x_0(t)$$

and if $t \in (t_{k-1}, t_k]$, we obtain

$$y_0(t) - x(t) = 1 - t_{k-1} - x(t) \ge 1 - t_{k-1} - (1 - t_k) = y_0(t) - x_0(t).$$

Hence

$$\|y_0 - x\| \ge \|y_0 - x_0\|$$

for each $x \in s(\Delta)$, which completes the proof of (5).

Now for each $y \in S(\Delta)$ we will find $x \in s(\Delta)$ such that

$$\|y_0 - x_0\| \geq \|y - x\| \geq d(y, s(\Delta)). \tag{6}$$

Let x be defined by

$$x(t) = \begin{cases} x_0(t), & y(t) \in [x_0(t), y_0(t)], \\ y(t), & y(t) \in [0, x_0(t)). \end{cases}$$

It is clear that

$$|y_0(t) - x_0(t)| \geq |y(t) - x_0(t)| = |y(t) - x(t)|$$

if $y(t) \in [x_0(t), y_0(t)]$ and

$$|y_0(t) - x_0(t)| \geq 0 = |y(t) - x(t)|$$

for $y(t) \in [0, x_0(t))$. Thus $\|y_0 - x_0\| \geq \|y - x\|$ and (6) holds.

By (5) and (6) we obtain

$$\begin{aligned} e\left(S(\Delta), s(\Delta)\right) = \|y_0 - x_0\| &= \sup_{t \in [0,1]} |y_0(t) - x_0(t)| \\ &= \max_{k \in \{1, \dots, n\}} t_k - t_{k-1} = \delta(\Delta) \\ &< \varepsilon \end{aligned}$$

and

$$h\left(S(\Delta, \tau), S(\Delta', \tau')\right) < 2\varepsilon,$$

which completes the proof.

References

[1] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. **580**, Springer-Verlag, Berlin – New York, 1977.

[2] F.S. De Blasi, *On the differentiability of multifunctions*, Pacific J. Math. **66** (1976), no. 1, 67-81.

[3] A. Dinghas, *Zum Minkowskischen Integralbegriff abgeschlossener Mengen*, Math. Z. **66** (1956), 173-188.

[4] M. Hukuhara, *Intégration des applications mesurables dont la valeur est un compact convexe*, Funkcial. Ekvac. **10** (1967), 205-223.

[5] B. Piątek, *On the Sincov functional equation*, Demonstratio Math. **38** (2005), no. 4, 875-881.

[6] M. Piszczek, *Integral representations of convex and concave set-valued functions*, Demonstratio Math. **35** (2002), no. 4, 727-742.

[7] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165-169.

[8] R. Urbański, *A generalization of the Minkowski–Rådström–Hörmander theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), no. 9, 709-715.

Institute of Mathematics
Silesian University of Technology
Kaszubska 23
PL-44-100 Gliwice
Poland
E-mail: b.piatek@polsl.pl

Received: 13 May 2007; final version: 14 July 2007;
available online: 9 November 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Małgorzata Wróbel

Lichawski-Matkowski-Miś theorem on locally defined operators for functions of several variables

Abstract. Let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$ and $C^m(D)$ be the space of functions $f|_D$ such that $f \in C^m(G)$. The representation formulas for locally defined operators mapping $C^m(D)$ into $C^0(D)$ and into $C^1(D)$ are given.

1. Introduction

For a real interval $I \subset \mathbb{R}$ and a nonnegative integer m, we denote by $C^m(I)$ the set of all m-times continuously differentiable functions $\varphi: I \longrightarrow \mathbb{R}$. An operator $K: C^m(I) \longrightarrow C^0(I)$ or $C^m(I) \longrightarrow C^1(I)$ is said to be locally defined if for every two functions $\varphi, \psi \in C^m(I)$ and for every open subinterval $J \subset I$ the relation $\varphi|_J = \psi|_J$ implies that $K(\varphi)|_J = K(\psi)|_J$. Answering a question posed by F. Neuman, the authors of [1] gave a representation formula for locally defined operators $K: C^m(I) \longrightarrow C^0(I)$. Namely, they proved that: *every locally defined operator* $K: C^m(I) \to C^0(I)$ *must be of the form*

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(m)}(x))$$

for a certain function $h: I \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$. Moreover, they proved that *every locally defined operator* $K: C^m(I) \longrightarrow C^1(I)$ *must be of the form*

$$K(\varphi)(x) = h(x, \varphi(x), \ldots, \varphi^{(m-1)}(x)).$$

In this paper we generalize this result showing that analogous representation theorems hold true for locally defined operators $K: C^m(D) \longrightarrow C^0(D)$ and $C^m(D)$ into $C^1(D)$, where D is a regular closed set in the open subspace $G \subset \mathbb{R}^n$ and $C^m(D)$ is the space of functions $f|_D$ such that $f \in C^m(G)$. The proofs of our theorems are similar in spirit to the proofs of Theorems 2 and 3 in [1].

AMS (2000) Subject Classification: 47H30.

2. Preliminaries

Let $\mathbb{N}_0$ be a set of nonnegative integers and let $\mathbb{N}_0^n := \prod_{i=1}^n \mathbb{N}_0$ for $n \in \mathbb{N}$. In this paper, for $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and $i = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, we put

$$\begin{aligned} |k| &:= k_1 + \ldots + k_n, \\ k! &:= (k_1!) \cdot \ldots \cdot (k_n!), \\ k + i &:= (k_1 + i_1, \ldots, k_n + i_n), \\ k - i &:= (k_1 - i_1, \ldots, k_n - i_n) \quad \text{for all } i \leq k, \end{aligned}$$

where the notation $i \leq k$ means that $i_s \leq k_s$ for every $s \in \{1, \ldots, n\}$.

Moreover, for $i = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we put

$$x^i := x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \qquad \text{and} \qquad \|x\| := \sqrt{\sum_{i=1}^n x_i^2}.$$

As a consequence of the Whitney Extension Theorem (cf. [2]) we get the following lemma.

Lemma 1
Let $B \subset \mathbb{R}^n$ be a compact set with only one cluster point $z \in \mathbb{R}^n$. Suppose that $m \in \mathbb{N}_0$ and

$$\{f^k \mid f^k : B \longmapsto \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \leq m\} \qquad \textit{where } f^{(0,\ldots,0)} = f$$

is a family of functions satisfying the condition

$$f^k(x) - \sum_{|i| \leq m-|k|} \frac{f^{k+i}(z)}{i!}(x-z)^i = o(\|x-z\|^{m-|k|}) \qquad \textit{as } x \to z \tag{1}$$

for all $x \in B$, $|k| \leq m$, $k \in \mathbb{N}_0^n$. If for some $\alpha > 0$,

$$x \neq y \implies \|x - y\| \geq \alpha \max\{\|x - z\|, \|y - z\|\}, \qquad x, y \in B,$$

then there exists a function g of the class C^m on $\mathbb{R}^n$ satisfying the condition

$$\frac{\partial^{|k|} g}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}(x) = f^k(x) \qquad \textit{for all } x \in B,\ k \in \mathbb{N}_0^n \textit{ and } |k| \leq m. \tag{2}$$

3. Locally defined operators mapping $C^m(D)$ into $C^0(D)$ and into $C^1(D)$

Let G be a nonempty and open set in the Euclidean space $\mathbb{R}^n$. By $C^m(G)$ we denote the space of m-times continuously differentiable functions on G.

Definition 1
Let G be an open set in the Euclidean space $\mathbb{R}^n$ and let $D \subset G$ be a regular closed set in the subspace G, i.e., $D = G \cap \operatorname{cl} \operatorname{int} D$. A function $f: D \longrightarrow \mathbb{R}$ is said to be of the *class* C^m *on* D if there exists a function $g \in C^m(G)$ such that $g|_D = f$, i.e.,

$$C^m(D) = \{f|_D : \ f \in C^m(G)\}.$$

Let $J_i \subset \mathbb{R}$, $i = 1, \ldots, n$, be open (closed) intervals. A set $J \subset \mathbb{R}^n$,

$$J = \prod_{i=1}^{n} J_i \,,$$

the Cartesian product of the intervals J_i, will be called an *open* (*closed*) *interval* in $\mathbb{R}^n$.

Now, we introduce the definition of locally defined operators of the type $K: C^m(D) \longrightarrow C^k(D)$.

Definition 2
Let $m, k \in \mathbb{N}_0$ and let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$. An operator $K: C^m(D) \longrightarrow C^k(D)$ is said to be *locally defined* if for every two functions $\varphi, \psi \in C^m(D)$ and for every open interval $J \subset \mathbb{R}^n$

$$\varphi|_{D\cap J} = \psi|_{D\cap J} \implies K(\varphi)|_{D\cap J} = K(\psi)|_{D\cap J} \,.$$

We shall need the following lemma.

Lemma 2 (cf. [3], Theorem)
Let $m, k \in \mathbb{N}_0$ and a closed interval $D \subset \mathbb{R}^n$ be fixed and let $K: C^m(D) \longrightarrow C^k(D)$ be a locally defined operator. Then for every $x_o \in D$, $\varphi, \psi \in C^m(D)$, if

$$\frac{\partial^{|j|}\varphi}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}(x_o) = \frac{\partial^{|j|}\psi}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}(x_o) \qquad \text{for all } j \in \mathbb{N}_0^n\,,\ |j| \le m,$$

then

$$\frac{\partial^{|i|}K(\varphi)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}(x_o) = \frac{\partial^{|i|}K(\psi)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}(x_o) \qquad \text{for all } i \in \mathbb{N}_0^n\,,\ |i| \le k.$$

Before formulating the main theorems we have to introduce the following notation. Let $m \in \mathbb{N}_0$ be fixed. Then

$$S(k) := \sum_{s=0}^{m-k} \binom{n+s-1}{s}$$

denotes the cardinality of the set of all partial derivatives of $m-k$ times continuously differentiable function $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$.

Theorem 1
Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$. If an operator $K\colon C^m(D) \longrightarrow C^0(D)$ is locally defined, then there exists a unique function $h\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x), \frac{\partial\phi}{\partial x_1}(x), \ldots, \frac{\partial\phi}{\partial x_n}(x), \ldots, \frac{\partial^m\phi}{\partial x_1^m}(x), \ldots, \frac{\partial^m\phi}{\partial x_n^m}(x)\right)$$

for all $\phi \in C^m(D)$ and $x \in D$.

Proof. The proof is based on the concept of Theorem 2 in [1]. In order to define a function $h\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ let us fix arbitrarily $z = (z_1, \ldots, z_n) \in D$ and $y_{(j_1,\ldots,j_n)} \in \mathbb{R}$ such that $j_1, \ldots, j_n \in \{0, \ldots, m\}$, $|j| \le m$. Let us take a polynomial

$$P_{z_1,\ldots,z_n,y_{(0,\ldots,0)},\ldots,y_{(0,\ldots,m)}}(x_1, \ldots, x_n)$$
$$:= \sum_{\substack{j_1,\ldots,j_n=0\\ j_1+\ldots+j_n\le m}}^{m} \frac{y_{(j_1,\ldots,j_n)}}{j_1!\ldots j_n!}(x_1 - z_1)^{j_1} \cdot \ldots \cdot (x_n - z_n)^{j_n}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n$$

and put

$$h(z_1, \ldots, z_n, y_{(0,\ldots,0)}, \ldots, y_{(0,\ldots,m)}) := K(P_{z_1,\ldots,z_n,y_{(0,\ldots,0)},\ldots,y_{(0,\ldots,m)}})(z_1, \ldots, z_n).$$

For any $\phi \in C^m(D)$, $j \in \mathbb{N}_0^n$ and $|j| \le m$

$$\frac{\partial^{|j|}\phi}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}(z_1, \ldots, z_n) = \frac{\partial^{|j|} P_{z_1,\ldots,z_n,\phi(z),\frac{\partial\phi}{\partial x_1}(z),\ldots,\frac{\partial^m\phi}{\partial x_n^m}(z)}}{\partial x_1^{j_1} \ldots \partial x_n^{j_m}}(z_1, \ldots, z_n).$$

Hence, by Lemma 2 for $|i| = 0$, we obtain

$$K(\phi)(z_1, \ldots, z_n) = K\big(P_{z_1,\ldots,z_n,\phi(z),\frac{\partial\phi}{\partial x_1}(z),\ldots,\frac{\partial^m\phi}{\partial x_n^m}(z)}\big)(z_1, \ldots, z_n)$$

and therefore

$$K(\phi)(z_1, \ldots, z_n) = h\left(z_1, \ldots, z_n, \phi(z), \frac{\partial\phi}{\partial x_1}(z), \ldots, \frac{\partial^m\phi}{\partial x_n^m}(z)\right).$$

Now, we prove the uniqueness of h. Let $h_1\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ be a function such that

$$K(\phi)(z_1, \ldots, z_n) = h_1\left(z_1, \ldots, z_n, \phi(z), \frac{\partial\phi}{\partial x_1}(z), \ldots, \frac{\partial^m\phi}{\partial x_n^m}(z)\right)$$

for all $\phi \in C^m(D)$ and $z = (z_1, \ldots, z_n) \in D$. In order to show that $h = h_1$, let us fix an arbitrary $(z_1, \ldots, z_n) \in D$ and $y_{(j_1,\ldots,j_n)} \in \mathbb{R}$, $j_1, \ldots, j_n \in \{0, \ldots, m\}$, $|j| \le m$.

According to the definitions of h_1 and h, we have

$$\begin{aligned} h_1(z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}) &= K(P_{z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}})(z_1,\dots z_n) \\ &= h(z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}), \end{aligned}$$

which completes the proof.

Corollary 1
Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and an open set $G \subset \mathbb{R}^n$ be fixed. If an operator $K\colon C^m(G) \longrightarrow C^0(G)$ is locally defined, then there exists a unique function $h\colon G \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x), \frac{\partial \phi}{\partial x_1}(x), \dots, \frac{\partial \phi}{\partial x_n}(x), \dots, \frac{\partial^m \phi}{\partial x_1^m}(x), \dots, \frac{\partial^m \phi}{\partial x_n^m}(x)\right)$$

for all $\phi \in C^m(G)$ and $x \in G$.

The following result may be proved in much the same way as Theorem 3 in [1].

Theorem 2
Let $m, n \in \mathbb{N}$ and let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$. If an operator $K\colon C^m(D) \longrightarrow C^1(D)$ is locally defined, then there exists a unique function $h\colon D \times \mathbb{R}^{S(1)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x), \dots, \frac{\partial^{m-1} \phi}{\partial x_1^{m-1}}(x), \dots, \frac{\partial^{m-1} \phi}{\partial x_n^{m-1}}(x)\right)$$

for all $\phi \in C^m(D)$ and $x = (x_1, \dots, x_n) \in D$.

Proof. By Theorem 1 there exists a unique function $h\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ such that for all $\phi \in C^m(D)$ and $(x_1, \dots, x_n) \in D$

$$\begin{aligned} &K(\phi)(x_1,\dots,x_n) \\ &= h\left(x_1,\dots,x_n,\phi(x_1,\dots,x_n),\dots,\frac{\partial^m \phi}{\partial x_1^m}(x_1,\dots,x_n),\dots,\frac{\partial^m \phi}{\partial x_n^m}(x_1,\dots,x_m)\right). \end{aligned}$$

In order to prove this theorem it is enough to show that for all $i \in \mathbb{N}_0^n$ such that $|i| = m$ we have

$$\frac{\partial h}{\partial y_i}(x_1,\dots,x_n,y_{(0,\dots,0)},\dots,y_{(m,0,\dots,0)},\dots,y_{(0,\dots,0,m)}) = 0. \tag{3}$$

Let us fix $x_o \in D$ and $y_i \in \mathbb{R}$ where $i \in \mathbb{N}_0^n$, $|i| \le m$ and let us choose an arbitrary i_0, $|i_0| = m$, and a real sequence $(y_{i_0,N})_{N=0}^{\infty}$ such that

$$y_{i_0,0} = y_{i_0}; \qquad y_{i_0,N} \neq y_{i_0}, \quad N \in \mathbb{N}; \qquad \lim_{N\to\infty} y_{i_0,N} = y_{i_0,0}.$$

Let ϕ_N, for every $N \in \mathbb{N}_0$, denotes the polynomial

$$\phi_N(x) := \sum_{\substack{|r|\le m \\ r\ne i_0}} \frac{y_r}{r!}(x-x_o)^r + \frac{y_{i_0,N}}{i_0!}(x-x_o)^{i_0}, \qquad x \in D.$$

Fix an $\varepsilon > 0$. Since all functions $K(\phi_N)$ are continuous, for all $N \in \mathbb{N}$ there exists $\delta_N > 0$ such that

$$\|x - x_o\| < \delta_N \;\Rightarrow\; |K(\phi_N)(x) - K(\phi_N)(x_o)| < \varepsilon |y_{i_0,N} - y_{i_0,0}|, \quad x \in D. \tag{4}$$

Take an arbitrary $\alpha > 0$ and choose a set $B = \{x_N : \; N \in \mathbb{N}_0\} \subset D$ satisfying all the conditions listed in Lemma 1 with $z = x_o$ and such that

$$\|x_N - x_o\| < \delta_N\,, \qquad N \in \mathbb{N} \tag{5}$$

and

$$\lim_{N\to\infty} \frac{y_{i_0,N} - y_{i_0,0}}{\|x_N - x_o\|} = \infty. \tag{6}$$

Now define functions $f^i\colon \mathbb{B} \longrightarrow \mathbb{R}$, $i \in \mathbb{N}_0^n$, $|i| \le m$, by the formula

$$f^i(x_N) := \phi_N^i(x_N), \qquad N \in \mathbb{N}_0\,.$$

First we show that the family $\{f^i \mid f^i\colon \mathbb{B} \longrightarrow \mathbb{R},\; i \in \mathbb{N}_0^n,\; |i| \le m\}$ fulfills (1) for all $i \in \mathbb{N}_0^n$ such that $i \le i_0$.

Since for all $N \in \mathbb{N}_0$

$$f^i(x_N) = \sum_{\substack{|r|\le m-|i| \\ r\ne i_0-i}} \frac{y_{i+r}}{r!}(x_N-x_o)^r + \frac{y_{i_0,N}}{(i_0-i)!}(x_N-x_o)^{i_0-i},$$

and

$$\sum_{|r|\le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N-x_o)^r = \sum_{\substack{|r|\le m-|i| \\ r\ne i_0-i}} \frac{y_{i+r}}{r!}(x_N-x_o)^r + \frac{y_{i_0,0}}{(i_0-i)!}(x_N-x_o)^{i_0-i},$$

we infer that

$$\begin{aligned}
\left| f^i(x_N) - \sum_{|r|\le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N-x_o)^r \right| &= \left| \frac{y_{i_0,N} - y_{i_0,0}}{(i_0-i)!}(x_N-x_o)^{i_0-i} \right| \\
&= \frac{|y_{i_0,N} - y_{i_0,0}|}{(i_0-i)!} \; |(x_N-x_o)^{i_0-i}| \\
&\le \frac{|y_{i_0,N} - y_{i_0,0}|}{(i_0-i)!} \|x_N - x_o\|^{|i_0-i|} \\
&= \frac{|y_{i_0,N} - y_{i_0,0}|}{(i_0-i)!} \|x_N - x_o\|^{m-|i|} \\
&= o(\|x_N - x_o\|^{m-|i|}).
\end{aligned}$$

In the second case, when $i \in \mathbb{N}_0^n$ is such that $|i| \le m$ does not satisfy the inequality $i \le i_0$, we have

$$f^i(x_N) = \sum_{|r| \le m-|i|} \frac{y_{i+r}}{r!}(x_N - x_o)^r = \sum_{|r| \le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N - x_o)^r$$

and therefore

$$f^i(x_N) - \sum_{|r| \le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N - x_o)^r = 0.$$

Thus the family $\{f^i \mid f^i\colon \mathbb{B} \longrightarrow \mathbb{R},\ i \in \mathbb{N}_0^n,\ |i| \le m\}$ fulfills (1) and according to Lemma 1 there exists a function $g \in C^m(\mathbb{R}^n)$ such that

$$\frac{\partial^{|i|} g}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(x_N) = \frac{\partial^{|i|} \phi_N}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(x_N), \qquad N \in \mathbb{N}_0,\ i \in \mathbb{N}_0^n,\ |i| \le m. \tag{7}$$

Hence and by (4), (5), (7) and Lemma 2 we have

$$\begin{aligned}
&\left| \frac{h(x_o, y_{(0,\dots,0)}, \dots, y_{i_0,N}, \dots, y_{(0,\dots,m)}) - h(x_o, y_{(0,\dots,0)}, \dots, y_{i_0,0}, \dots, y_{(0,\dots,m)})}{y_{i_0,N} - y_{i_0,0}} \right| \\
&\quad = \left| \frac{K(\phi_N)(x_o) - K(\phi_0)(x_o)}{y_{i_0,N} - y_{i_0,0}} \right| \\
&\quad \le \left| \frac{K(\phi_N)(x_N) - K(\phi_N)(x_o)}{y_{i_0,N} - y_{i_0,0}} \right| + \left| \frac{K(\phi_N)(x_N) - K(\phi_0)(x_o)}{y_{i_0,N} - y_{i_0,0}} \right| \\
&\quad \le \varepsilon + \left| \frac{K(g)(x_N) - K(g)(x_o)}{y_{i_0,N} - y_{i_0,0}} \right| \\
&\quad = \varepsilon + \frac{|K(g)(x_N) - K(g)(x_o)|}{\|x_N - x_o\|} \cdot \frac{\| x_N - x_o\|}{|y_{i_0,N} - y_{i_0,0}|}.
\end{aligned}$$

Since $K(g) \in C^1(D)$, we conclude that

$$\lim_{N \to \infty} \frac{|K(g)(x_N) - K(g)(x_o)|}{\|x_N - x_o\|} < \infty.$$

Hence and by (6) we obtain (3) for $i = i_0 \in \mathbb{N}_0^n$ such that $|i_0| = m$ and the proof is completed.

Corollary 2

Let $m, n \in \mathbb{N}$ and an open set $G \subset \mathbb{R}^n$ be fixed. If an operator $K\colon C^m(G) \longrightarrow C^1(G)$ is locally defined, then there exists a unique function $h\colon G \times \mathbb{R}^{S(1)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x), \dots, \frac{\partial^{m-1}\phi}{\partial x_1^{m-1}}(x), \dots, \frac{\partial^{m-1}\phi}{\partial x_n^{m-1}}(x)\right)$$

for all $\phi \in C^m(G)$ and $x \in G$.

Acknowledgements

I would like to thank the referee for helpful comments and suggestions.

References

[1] K. Lichawski, J. Matkowski, J. Miś, *Locally defined operators in the space of differentiable functions*, Bull. Polish Acad. Sci. Math. **37** (1989), no. 1-6, 315-325.

[2] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63-89.

[3] M. Wróbel, *Some consequences of locally defined operators*, in: Proceedings of the XII$^{\mathrm{th}}$ Czech–Polish–Slovak Mathematical School, Faculty of Education of University J.E. Purkyně in Ústi nad Labem, 2005, 82-86.

Institute of Mathematics
and Informatics
Jan Dlugosz University
Armii Krajowej 13/15
PL-42-218 Częstochowa
Poland
E-mail: m.wrobel@ajd.czest.pl

Received: 20 November 2006; final version: 30 May 2007;
available online: 9 November 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

V. Kulik, E. Tkocz-Piszczek

Sign-changing Lyapunov functions in linear extensions of dynamical systems

Abstract. In this note we consider sets of linear extensions of dynamical systems on a torus. We examine regularity of the systems by means of a given sign-changing Lyapunov function. The main result of the paper is to give conditions of regularity for the set of differential equations with degenerated matrix of coefficients.

Let us consider a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ \dfrac{dy}{dt} = A(x)y, \end{cases} \tag{1}$$

where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_n)$, $A(x)$ is a square, n-dimensional matrix, which elements are periodic with period 2π, continuous with respect to each variable x_j , $j = 1, \dots, k$, it means it is specified on an k-dimensional torus T_k . The set of all such functions which are continuous and periodic with period 2π with respect to each variable x_j , $j = 1, \dots, k$, is denoted by $C^0(T_k)$. We assume that the function $f(x)$ satisfies the Lipschitz inequality $||f(x) - f(\bar{x})|| \leq L||x - \bar{x}||$ for all $x, \bar{x} \in T_k$, $L = \text{const} > 0$, where $||y||^2 = \langle y, y\rangle$ is the Euclidean norm in the space $\mathbb{R}^n$, $y \in \mathbb{R}^n$, and $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$ is an inner product in $\mathbb{R}^n$. We denote by $C_{Lip}(T_k)$ a space of functions $f(x) \in C^0(T_k)$, which satisfy the Lipschitz inequality. It follows that $f(x) \in C_{Lip}(T_k)$ and $A(x) \in C^0(T_k)$. Let us also denote by $||A|| = \max\limits_{||y||=1} ||Ay||$ the norm of a $n\times n$-dimensional matrix A taken as an operator. In $C^0(T_k)$ we distinguish a subspace $C'(T_k; f)$ of functions $F(x)$ such that the superposition $F(x(t; x))$ is continuously differentiable with respect to $t \in \mathbb{R}$, where $x(t; x)$ is a solution to the Cauchy problem

AMS (2000) Subject Classification: 34C99, 37C40, 37C99.

$$\frac{dx}{dt} = f(x), \qquad x\big|_{t=0}\ , \ \forall x \in T_k\,.$$

We define

$$\dot F(x) \stackrel{\mathrm{df}}{=} \frac{dF(x(t;x))}{dt}\bigg|_{t=0} \qquad \text{for } \dot F(x) \in C^0(T_k).$$

In $C^0(T_k)$ we also distinguish a subspace $C^1(T_k)$ of functions $F(x)$, which have continuous first derivatives with respect to each variable x_j, $j = 1, \ldots, k$. Let $u(\varphi) \in C^1(T_k)$. Then $\dot u(\varphi) = \sum_{j=1}^{k} \frac{\partial u(\varphi)}{\partial \varphi} f_j(\varphi) = \frac{\partial u}{\partial \varphi} f(\varphi)$.

Definition 1

We say that the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ \dfrac{dy}{dt} = A(x)y + h(x), \qquad h(x) \in C^0(T_k) \end{cases} \tag{2}$$

possesses a torus

$$y = u(x),$$

if $u(x) \in C'(T_k; f)$ and the identity

$$\dot u(x) \equiv A(x)u(x) + h(x), \qquad \forall x \in T_k$$

holds.

Example

Let us consider a system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = 1, \\ \dfrac{dx_2}{dt} = \sqrt{3}, \\ \dfrac{dy}{dt} = (\gamma - \sin(x_1 + x_2))y + h(x_1, x_2), \end{cases}$$

where $\gamma = \mathrm{const} \in \mathbb{R}$, $h(x_1, x_2) \in C^1(T_2)$. An invariant torus for the system has the following form.

I. Case $\gamma > 0$.

$$\begin{aligned} y &= u(x_1, x_2) \\ &= -\int_0^\infty e^{-\gamma\tau - \frac{1}{1+\sqrt{3}}[\cos((1+\sqrt{3})\tau + x_1 + x_2) - \cos(x_1 + x_2)]} h(\tau + x_1, \sqrt{3}\tau + x_2)\, d\tau. \end{aligned}$$

II. Case $\gamma < 0$.

$$y = u(x_1, x_2)$$
$$= \int_{-\infty}^{0} e^{\gamma\tau + \frac{1}{1+\sqrt{3}}[\cos((1+\sqrt{3})\tau + x_1 + x_2) - \cos(x_1+x_2)]} h(\tau + x_1, \sqrt{3}\tau + x_2)\, d\tau.$$

III. Case $\gamma = 0$. The invariant torus for the system fails to exists for every $h(x) \in C^1(T_2)$. For example, when $h \equiv 2$, the torus does not exist.

Definition 2
Let $C(x)$ be an $n \times n$-dimensional continuous matrix, $C(x) \in C^0(T_k)$. Then the function $G_0(\tau, x)$ defined by

$$G_0(\tau, x) = \begin{cases} \Omega_\tau^0(x) C(x(\tau, x)), & \tau \leq 0, \\ \Omega_\tau^0(x)[C(x(\tau, x)) - I_n], & \tau > 0, \end{cases} \tag{3}$$

which satisfies the estimate

$$||G_0(\tau, x)|| \leq K e^{-\gamma |x|}, \tag{4}$$

where K and γ are positive constants, is called the *Green function* of the invariant torus for the system (1).

$\Omega_x^t(x)$ is the fundamental matrix of the solutions of the system $\frac{dy}{dt} = A(x(t; x))y$ which takes the value of the n-dimensional identity matrix for $t = x$ $\Omega_x^t(x)\big|_{t=x} = I_n$.

If the Green function (3) exists, then for every vector function $h(x) \in C^0(T_k)$ an invariant torus for the system (2) exists and it is defined by the formula

$$y = u(x) = \int_{-\infty}^{\infty} G_0(\tau, x) h(x(\tau, x))\, d\tau.$$

Definition 3
We say that the system (1) is *regular* if there exists a unique Green function (3) satisfying (4).

It is obvious [3], that the system (1) is regular when the square form

$$V = \langle S_0(x) y, y \rangle, \tag{5}$$

with the symmetric matrix $S_0(x) \in C^1(T_k)$, exists and its derivative along the solutions of the system (1) is positive definite:

$$\dot{V} = \left\langle \left[\frac{\partial S_0(x)}{\partial x} f(x) + S_0(x) A(x) + A^T(x) S_0(x) \right] y, y \right\rangle \geq \varepsilon ||y||^2, \tag{6}$$

$\varepsilon = \text{const} > 0$, and the matrix $S_0(x)$ satisfies the condition

$$\det S_0(x) \neq 0, \qquad \forall x \in T_k.$$

Dealing with problems of regularity of systems we find out issues which have not been touched upon in researches. We shall prove the existence of the regular system (1) for which $\det A(x) \equiv 0$ for all $x \in T_k$. The next problem is the analysis of right-hand sides of the system (1) for the which the derivative of the square form (5) along the solutions of the system is positive definite.

First of all, let us notice that the inequality (6) does not change for small perturbations of the vector function $f(x)$ and the matrix $A(x)$. We will show that in the right-hand side of the system (1), $f(x)$ can be substituted by any different function $b(x) \in C_{Lip}(T_k)$ and at the same time the matrix $A(x)$ can be chosen in such a way that the derivative of the square form (5) along the solutions of the system is positive definite. Thus the matrix $A(x)$ has the form

$$A(x) = S_0^{-1}(x) \left[B(x) + M(x) - 0.5 \frac{\partial S_0(x)}{\partial x} b(x) \right], \tag{7}$$

where $B(x), M(x) \in C^0(T_k)$ are any matrices which satisfy

$$B^T(x) \equiv B(x), \qquad \langle B(x)x, x \rangle \geq \lambda ||x||^2, \; \lambda = \text{const} > 0, \tag{8}$$

$$M^T(x) \equiv -M(x). \tag{9}$$

Let us check whether it is true. We consider the left-hand side of (6), substituting the function $f(x)$ with any vector function $b(x)$. We also assume the form of the matrix $A(x)$ to be like the one in (7):

$$\begin{aligned} \frac{\partial S_0(x)}{\partial x} f(x) + S_0(x)A(x) + A^T(x)S_0(x) = & \frac{\partial S_0(x)}{\partial x} b(x) + B(x) + M(x) \\ & - 0.5 \frac{\partial S_0(x)}{\partial x} b(x) + B(x) + M^T(x) \\ & - 0.5 \frac{\partial S_0(x)}{\partial x} b(x) \\ = & \, 2B(x). \end{aligned}$$

It follows that, when (8) is fulfilled, the inequality (6) is fulfilled for $\varepsilon = 2\lambda$. Then we get the following lemma.

Lemma
To any non-degenerate matrix $S_0(x) \in C^1(T_k)$ there corresponds the set of regular systems

$$\begin{cases} \dfrac{dx}{dt} = b(x), \\ \dfrac{dy}{dt} = S_0^{-1}(x) \left[B(x) + M(x) - 0.5 \dfrac{\partial S_0(x)}{\partial x} b(x) \right] y, \end{cases} \tag{10}$$

where $b(x)$ *is any vector function,* $b(x) \in C_{Lip}(T_k)$*,* $B(x), M(x)$ *are any continuous matrices satisfying* (8) *and* (9)*.*

Remark 1
The derivative of the square form (5) with the symmetric non-degenerate matrix $S_0(x) \in C^1(T_k)$ along the solutions of the system (10) has the form $\dot{V} = 2\langle By, y\rangle$.

Theorem 1
Systems (1)*, in which* $\det A(x) \equiv 0$ *for all* $x \in T_k$ *, exist.*

Proof. We define

$$S_0(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}, \qquad \psi = x_1 + x_2 + x_3 + \ldots + x_k\,. \tag{11}$$

We shall consider the system (10) of the form

$$\begin{cases} \dfrac{dx_i}{dt} = \omega_i\,, \\ \dfrac{dy}{dt} = S_0^{-1}(\psi)\Big[B + M - \dfrac{1}{2}\cdot\dfrac{dS_0(\psi)}{d\psi}\displaystyle\sum_{j=1}^{k}\omega_j\Big]y, \end{cases} \qquad \omega_i = \text{const}, \tag{12}$$

where $i = 1, \ldots, k$ and $y \in \mathbb{R}^2$. Let B and M be constant matrixes

$$B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad b > 0, \qquad M = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}. \tag{13}$$

The derivative of the non-degenerate square form $V = y_1^2\cos\psi + 2y_1y_2 sin\psi - y_2^2\cos\psi$ along the solutions of the system (12) is positive definite, hence the system (12) is regular. Taking $\omega = \sum\omega_j$ we obtain

$$A(x) = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}\left\{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} - 0.5\omega\begin{pmatrix} -\sin\psi & \cos\psi \\ \cos\psi & \sin\psi \end{pmatrix}\right\}$$

$$= \begin{pmatrix} b\cos\psi + m\sin\psi & b\sin\psi - m\cos\psi - 0.5\omega \\ b\sin\psi - m\cos\psi + 0.5\omega & -b\cos\psi - m\sin\psi \end{pmatrix}. \tag{14}$$

We have $\det A(x) = -b^2 - m^2 + 0.25\omega^2$. Therefore, the identity $\det A(x) \equiv 0$ holds when $\omega^2 = 4(b^2 + m^2)$.

Remark 2
Using change of variables

$$y = \begin{pmatrix} \cos\frac{\psi}{2} & \sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & -\cos\frac{\psi}{2} \end{pmatrix} z, \qquad z \in \mathbb{R}^2,$$

the system (12) with matrices (11) and (13) can be transformed into the following system with constant coefficients

$$\begin{cases} \dfrac{dx_i}{dt} = \omega_i\,, \\ \dfrac{dz_1}{dt} = bz_1 + mz_2\,, \\ \dfrac{dz_2}{dt} = mz_1 - bz_2\,. \end{cases}$$

Remark 2 confirms once again that the system (1) is a regular one, because eigenvalues λ_i of the matrix of coefficients $\begin{pmatrix} b & m \\ m & -b \end{pmatrix}$ for the given system satisfy the condition $\operatorname{Re}\lambda_i = \lambda_i \neq 0$.

Remark 3
If the Green function (3) with the estimate (4) exists, then the function

$$G_t(\tau, x) = \begin{cases} \Omega_\tau^t C(x(\tau, x)), & \tau \le t, \\ \Omega_\tau^t [C(x(\tau, x)) - I_n], & \tau > t \end{cases}$$

is called (cf. [3]) the Green function of the problem of the bounded solutions of the system $\frac{dy}{dt} = A(x(t;x))y$. It means that for any function $h(t)$, which is continuous and bounded, the system

$$\frac{dy}{dt} = A(x(t;x))y + h(t), \qquad \forall x \in T_k$$

has the unique bounded solution

$$y = \int_{-\infty}^{\infty} G_t(\tau, x) h(\tau)\, d\tau.$$

Based on the previous considerations let us note, that the linear system which corresponds to the system (12) with the matrix $A(2\omega t)$ given by (14) after replacing ω by 2ω can be written in the form

$$\begin{cases} \dot{y}_1 = (b\cos 2\omega t + m\sin 2\omega t)y_1 + (-m\cos 2\omega t + b\sin 2\omega t - \omega)y_2\,, \\ \dot{y}_2 = (-m\cos 2\omega t + b\sin 2\omega t + \omega)y_1 + (-b\cos 2\omega t - m\sin 2\omega t)y_2 \end{cases} \tag{15}$$

with constants $\omega, m, b \in \mathbb{R}$, $(b \neq 0)$. Let us note that $\det A(2\omega t) \equiv 0$ when the condition

$$\omega^2 = b^2 + m^2 \tag{16}$$

holds. The derivative of the non-degenerate square form $V = y_1^2 \cos \omega t + 2y_1 y_2 \sin \omega t - y_2^2 \cos \omega t$ along the solutions of the system is positive definite, thus the system is exponentially dichotomous in $\mathbb{R}$ (cf. [1], [3]). It means that the non-homogenous system

$$\begin{cases} \dot{y}_1 = (b \cos 2\omega t + m \sin 2\omega t) y_1 + (-m \cos 2\omega t + b \sin 2\omega t - \omega) y_2 \\ \qquad + h_1(t), \\ \dot{y}_2 = (-m \cos 2\omega t + b \sin 2\omega t + \omega) y_1 + (-b \cos 2\omega t - m \sin 2\omega t) y_2 \\ \qquad + h_2(t) \end{cases} \tag{17}$$

has a unique bounded solution in $\mathbb{R}$ for any vector function $h(t)$ which is continuous and bounded in $\mathbb{R}$.

Since we want to write down the solution of the system (17), we simplify the system (15). On the basis of Remark 2 we use the change of variables

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t \\ \sin \omega t & -\cos \omega t \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and the system (15) results in the system with constant coefficients

$$\begin{cases} \dot{z}_1 = b z_1 + m z_2 \,, \\ \dot{z}_2 = m z_1 - b z_2 \,. \end{cases} \tag{18}$$

Then another change of variables in the system (18) can be used:

$$z = Tr,$$

where

$$T = \begin{cases} \begin{pmatrix} \omega + b & -m \\ m & \omega + b \end{pmatrix}, & b > 0, \\ \begin{pmatrix} m & b - \omega \\ \omega - b & m \end{pmatrix}, & b < 0 \end{cases}$$

and we obtain the system with separated variables

$$\begin{cases} \dot{r}_1 = \omega r_1 \,, \\ \dot{r}_2 = \, - \omega r_2 \,. \end{cases}$$

Therefore, the bounded solution of the system (17) has the form

$$y = y^*(t) = L(t) T \int_{-\infty}^{\infty} G(t, \tau) T^{-1} L^{-1}(\tau) h(\tau) \, d\tau,$$

where

$$G(t,\tau) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \exp\{-\omega(t-\tau)\}, & \tau \le t, \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \exp\{\omega(t-\tau)\}, & \tau > t, \end{cases}$$

$$L(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ \sin\omega t & -\cos\omega t \end{pmatrix}.$$

Under the condition (16) the system (17) can be transformed into the scalar equation

$$\begin{aligned} &\left(\frac{dy_1}{dt} - h_1(t)\right)[m\cos(2\omega t) - b\sin(2\omega t) - \omega] \\ &\quad + \left(\frac{dy_2}{dt} - h_2(t)\right)[b\cos(2\omega t) + m\sin(2\omega t)] = 0. \end{aligned} \tag{19}$$

Remark 4
The equation (19) can be transformed into the form

$$\left(\frac{dy_1}{dt} - \bar{h}_1(t)\right)\sin t + \left(\frac{dy_2}{dt} - \bar{h}_2(t)\right)\cos t = 0, \tag{20}$$

where $\bar{h}_i(t) = \frac{1}{\omega} h_i\left(\frac{1}{\omega}t - \frac{\Delta+\pi}{2\omega}\right)$, $i = 1, 2$, $\cos\Delta = \frac{m}{\omega}$, $\sin\Delta = \frac{m}{\omega}$.

Now we consider (20) as a separate equation and we obtain that, apart from the solution $y = y^*(t) = (y_1^*(t), y_2^*(t))$, there exists the whole set of bounded solutions.

Remark 5
In the system (10) the variable $\xi \in T_l$ can be added to the variable x. Then we consider the system

$$\begin{cases} \dfrac{dx}{dt} = b(x,\xi), \\ \dfrac{d\xi}{dt} = \bar{b}(x,\xi), \\ \dfrac{dy}{dt} = S_0^{-1}\left[B(x,\xi) + M(x,\xi) - 0.5\dfrac{\partial S_0(x)}{\partial x} b(x,\xi)\right] y, \end{cases} \tag{21}$$

where $b(x,\xi), \bar{b}(x,\xi) \in C_{Lip}(T_{k+l})$ are any vector functions and for matrices $B(x,\xi), M(x,\xi) \in C^0(T_k \times T_l)$ identities $B^T \equiv B$, $M^T \equiv -M$ hold. The derivative of the square form (5) along the solutions of (21) has the form $\dot{V} = 2\langle B(x,\xi)y, y\rangle$.

Remark 6
If the derivative of the non-degenerate square form (5) along the solutions of the system

$$\begin{cases} \dfrac{dx}{dt} = b(x,\xi), \\ \dfrac{d\xi}{dt} = \bar{b}(x,\xi), \\ \dfrac{dy}{dt} = P(x,\xi)y, \end{cases}$$

where $x \in T_k$, $\xi \in T_l$, is positive definite, then

$$P(x,\xi) = S_0^{-1}(x)\left[B(x,\xi) + M(x,\xi) - 0.5\frac{\partial S_0(x)}{\partial x}b(x,\xi)\right], \qquad (22)$$

where the matrix $B(x,\xi)$ is symmetric positive definite and the matrix $M(x,\xi)$ is skew-symmetric.

If the derivative of the square form (5) along the solutions of the system (21) is positive definite, then

$$\dot{V} = \left\langle \left[\frac{\partial S_0(x)}{\partial x}B(x,\xi) + S_0(x)P(x,\xi) + P^T(x,\xi)S_0(x)\right]y, y\right\rangle \geq \varepsilon||y||^2,$$

$\varepsilon = \text{const} > 0$. Let us take matrices B and M of the following forms:

$$B(x,\xi) = 0.5\left[\frac{\partial S_0(x)}{\partial x}B(x,\xi) + S_0(x)P(x,\xi) + P^T(x,\xi)S_0(x)\right], \qquad (23)$$

$$M(x,\xi) = 0.5\left[S_0(x)P(x,\xi) - P^T(x,\xi)S_0(x)\right]. \qquad (24)$$

Then, above-mentioned matrixes (23) and (24) satisfy conditions (8) and (9) ($\lambda = \frac{\varepsilon}{2}$) and the equality (22) also holds.

Remark 7
If for the non-degenerate square form (5) we consider the set of systems (1), where the derivative of the form along the solutions of these systems is positive definite, we are able only to increase the number of variables x. Decreasing the number of variables x is not always possible.

This is confirmed by the following example. We consider the matrix $S_0(\psi)$ of the form (11). Let $\tilde{x} = (x_2, \ldots, x_k)$, $x = (x_1, \tilde{x})$. Let us assume that functions $f_j(\tilde{x})$ with respect to $\tilde{x}$, $f_j(\tilde{x}) \in C_{Lip}(T_{k-1})$, $j = 2, \ldots, k$ exist and, moreover, matrices $A(\tilde{x})$ for which inequalities

$$\left\langle\left[\sum_{j=2}^{k}\frac{\partial S_0(\psi)}{\partial x_j}f_j(\tilde{x}) + S_0(\psi)A(\tilde{x}) + A^T(\tilde{x})S_0(\psi)\right]y, y\right\rangle \geq \varepsilon||y||^2, \qquad (25)$$

$$\varepsilon = \text{const} > 0,\ \psi = x_1 + x_2 + x_3 + \ldots + x_k\,.$$

hold, exist. Taking into account the identities

$$S_0(\psi)\Big|_{x_1=0} \equiv -S_0(\psi)\Big|_{x_1=\pi},$$
$$\frac{\partial S_0(\psi)}{\partial x_j}\Bigg|_{x_1=0} \equiv -\frac{\partial S_0(\psi)}{\partial x_j}\Bigg|_{x_1=\pi} \qquad j=2,\ldots,k,$$

we obtain a contradiction to the formula (25):

$$\begin{aligned}&\left\langle\left[\sum_{j=2}^{k}\frac{\partial S_0(\psi)}{\partial x_j}f_j(\tilde{x}) + S_0(\psi)A(\tilde{x}) + A^T(\tilde{x})S_0(\psi)\right]y,y\right\rangle\Bigg|_{x_1=0}\\ &\equiv -\left\langle\left[\sum_{j=2}^{k}\frac{\partial S_0(\psi)}{\partial x_j}f_j(\tilde{x}) + S_0(\psi)A(\tilde{x}) + A^T(\tilde{x})S_0(\psi)\right]y,y\right\rangle\Bigg|_{x_1=\pi}\\ &\geq \varepsilon||y||^2.\end{aligned}$$

Let us take note of the fact that because of the form of the matrix (11) matrices $A(\tilde{x})$ are also continuous with respect to $\tilde{x}$ variables, thus a smaller number of variables than x, for which the inequality

$$\big\langle[S_0(\psi)A(\tilde{x}) + A^T(\tilde{x})S_0(\psi)]y,y\big\rangle \geq \varepsilon||y||^2, \qquad \varepsilon = \text{const} > 0$$

holds, does not exist. Let $2n\times 2n$-dimensional matrices $B(x), M(x) \in C^0(T_k)$ have the following forms:

$$B(x) = \begin{bmatrix} B_1(x) & B_{12}(x)\\ B_{12}^T(x) & B_2(x)\end{bmatrix}, \qquad M(x) = \begin{bmatrix} 0 & M(x)\\ -M^T(x) & 0\end{bmatrix}. \tag{26}$$

THEOREM 2
Let $B(x), M(x) \in C^0(T_k)$ be of the form (26) *and satisfy conditions* (8) *and* (9). *Then the system of equations*

$$\begin{cases}\dfrac{dy_1}{dt} = \big[-SB_1\sin\psi + \big[B_{12}^T(x) - M^T(x)\big]\,y_1\big] + \\ \qquad + \Bigg[-S\,[B_{12}(x) + M(x)]\sin\psi \\ \qquad + \Bigg(B_2(x) - 0.5S\Bigg(\displaystyle\sum_{j=1}^{k} f_j(x)\Bigg)\cos\psi\Bigg)\Bigg]y_2\,,\\ \dfrac{dy_2}{dt} = B_1(x)y_1 + [B_{12}(x) + M(x)]\,y_2\,,\\ \dfrac{dx}{dt} = f(x),\\ \psi = \displaystyle\sum_{j=1}^{k} x_j\end{cases} \tag{27}$$

is regular for any vector function $f(x) \in C_{Lip}(T_k)$ and any symmetric constant matrix S.

Proof. Let us consider the square forms

$$V = 2\langle y_1, y_2\rangle + \langle Sy_2, y_2\rangle \sin\psi, \qquad y_1, y_2 \in \mathbb{R}^n,\ \psi = x_1 + \ldots + x_k\,, \tag{28}$$

where the matrix S is $n\times n$-dimensional, constant and symmetric.

The matrix

$$S_0(x) = \begin{bmatrix} 0 & I_n \\ I_n & S\sin\psi \end{bmatrix}, \qquad \psi = \sum_{j=1}^{k} x_j\,,$$

which corresponds to the form (28), is non-degenerate. Obviously, the inverse matrix has the form

$$S_0^{-1}(x) = \begin{bmatrix} -S\sin\psi & I_n \\ I_n & 0 \end{bmatrix}.$$

We determine the matrix (7), when $b(x) = f(x)$:

$$S_0^{-1}(x)\left[B(x) + M(x) - 0.5\frac{\partial S_0(x)}{\partial x} f(x)\right]$$

$$= \begin{bmatrix} -S\sin\psi + (B_{12}^T - M^T) & -S(B_{12}+M)\sin\psi + \left(B_2 - 0.5S\left(\sum_{j=1}^{k} f_j\right)\cos\psi\right) \\ B_1 & B_{12}+M \end{bmatrix}.$$

Therefore the derivative of the square form (28) along the solutions of the system (27) is positive definite. Thus the system (27) is regular (cf. [3]).

References

[1] W.A. Coppel, *Dichotomies and Lyapunov functions*, J. Differential Equations **52** (1984), no. 1, 58-65.

[2] S. Elaydi, O. Hájek, *Exponential trichotomy of differential systems*, J. Math. Anal. Appl. **129** (1988), no. 2, 362-374.

[3] Yu.A. Mitropolsky, A.M. Samoilenko, V.L. Kulik, *Dichotomies and stability in nonautonomous linear systems*, Stability and Control: Theory, Methods and Applications **14**, Taylor & Francis, London, 2003.

Department of Mathematics
Silesian University of Technology
Kaszubska 23
PL-44-100 Gliwice, Poland
E-mail: Ewa.Tkocz@polsl.pl
E-mail: Wiktor.Kulyk@polsl.pl

Received: 10 May 2007; final version: 25 July 2007; available online: 9 December 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Ondřej Moc

Gauss's definition of the gamma function

Abstract. The present paper gives a historical account on extending the factorial function to complex numbers by Gauss.

A short excursion into history of complex numbers

Complex numbers (more exactly, square roots of negative numbers) have been used since the mid of the 16th century in mathematics. These numbers appeared when Italian mathematicians tried to solve polynomial equations of higher degrees. The use of complex numbers sometimes led to certain obscurities. A change of the perception of the complex numbers came around the turn of the 18th and 19th century when Karl Friedrich Gauss (1777-1855) published several papers where he used an idea of complex numbers.

Further progress came with representation of complex numbers by points or vectors in the plane. This idea occurred for the first time in the works of Caspar Wessel (1745-1818) and Jean Robert Argand (1768-1822); Argand introduced the term textitle module for an absolute value of the complex numbers. Both Wessel's and Argand's articles were written probably independently and these works have never received general awareness. The geometric interpretation of complex numbers was completely accepted when Gauss wrote his treatise *Theoria residuorum biquadraticorum* (Theory of the biquadratic residues) (1831), see [1].

In 1837 William Rowan Hamilton (1805-1865) introduced complex numbers as ordered pairs of real numbers. In 1847 Louis Augustin Cauchy (1789-1857) presented an algebraic definition of complex numbers.

A Gauss's letter to Bessel

On 21st November 1811 Karl Friedrich Gauss wrote a letter to Friedrich Wilhelm Bessel (1784-1846) about his development on general factorials, see [3]. Gauss wrote the following text in that letter (authors translation).

AMS (2000) Subject Classification: 01A55.

Thus the product

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot x = \prod x$$

is the function that, in my opinion, must be introduced into Calculus, (...). But if one wants to avoid countless Kramp's paradigms and paradoxes and contradictions, $1 \cdot 2 \cdot 3 \cdot \ldots \cdot x$ must not be used as the definition of $\prod x$, because such a definition has a precise meaning only when x is an integer; rather, one must start with a more general definition, which is applicable even to imaginary values of x, of which that one is a special case. I have chosen the following one

$$\prod x = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \cdot k^x}{(x+1)(x+2)(x+3) \cdot \ldots \cdot (x+k)},$$

where k tends to infinity.

Let us remark that Christian Kramp (1760-1826) was one of the mathematicians who sought a general rule for non-integer values of factorials. He introduced so-called "numerical factorial" by the form

$$a^{\frac{b}{c}} = a(a+c)(a+2c) \cdot \ldots \cdot (a+(b-1)c) \qquad (1)$$

in the book [4]. The product on the right-hand side of (1) was studied in the first half of the 19th century under the name "analytic factorial". In 1856 Karl Weierstrass (1815-1897) finished these activities, when he demonstrated nonsense resulting from that definition, see [8]. Moreover, Kramp was the first mathematician who used the notation $n!$ for n-factorials in the book [5].

Gauss acquired his knowledge about the function $\prod x$ during an investigation of properties of the hypergeometric series. The generalized hypergeometric function

$${}_pF_q(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p, x)$$

is defined by the sum of a hypergeometric series, i.e., series $\sum_{k=0}^{\infty} a_k x^k$ for which $a_0 = 1$ and the ratio of consecutive terms $\frac{a_{k+1}}{a_k}$ can be expressed as the fraction

$$\frac{a_{k+1}}{a_k} = \frac{(\alpha_1+k)(\alpha_2+k) \cdot \ldots \cdot (\alpha_p+k)}{(k+1)(\beta_1+k)(\beta_2+k) \cdot \ldots \cdot (\beta_q+k)} x.$$

If $p = 2$ and $q = 1$, we get *Gauss's hypergeometric function* ${}_2F_1(\alpha, \beta, \gamma, x)$, which is the sum of the series

$$1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \ldots \qquad (2)$$

Gauss dealt with that series in [2]. He stated a condition for the convergence of (2) in the terms of the coefficients α, β, γ, x, which is described in the following theorem.

THEOREM

Let $x \in \mathbb{C}$ and $\gamma \in \mathbb{C}\backslash\{0,-1,-2,-3,\ldots\}$. If $|x|<1$, then the series (2) *is convergent. If $|x|=1$, then the series* (2) *is convergent if and only if $|\gamma-\alpha-\beta|>0$ holds. In case of $|x|>1$, the series* (2) *is divergent.*

The introduction to the theory of the gamma function

Gauss derived a following formula for hypergeometric series

$$F(\alpha,\beta,\gamma,1)=\frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}F(\alpha,\beta,\gamma+1,1). \tag{3}$$

Then he generalized equation (3) into the form

$$\begin{aligned} &F(\alpha,\beta,\gamma,1)\\ &=\frac{(\gamma-\alpha)(\gamma+1-\alpha)\cdot\ldots\cdot(\gamma+k-1-\alpha)}{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+k-1)}\\ &\quad\times\frac{(\gamma-\beta)(\gamma+1-\beta)\cdot\ldots\cdot(\gamma+k-1-\beta)}{(\gamma-\alpha-\beta)(\gamma+1-\alpha-\beta)\cdot\ldots\cdot(\gamma+k-1-\alpha-\beta)}\;F(\alpha,\beta,\gamma+k,1), \end{aligned} \tag{4}$$

where $k\in\mathbb{N}$. This recurrence formula became a starting point for his next investigation about the gama function.

For $k\in\mathbb{N}$, $z\in\mathbb{C}$, Gauss introduced the function $\prod(k,z)$ by

$$\prod(k,z)=\frac{1\cdot2\cdot3\cdot\ldots\cdot k}{(z+1)(z+2)(z+3)\cdot\ldots\cdot(z+k)}k^z.$$

A simple calculation shows that the expression

$$\frac{\prod(k,\gamma-1)\cdot\prod(k,\gamma-\alpha-\beta-1)}{\prod(k,\gamma-\alpha-1)\cdot\prod(k,\gamma-\beta-1)}$$

can be reduced to the fraction

$$\begin{aligned} &\frac{(\gamma-\alpha)(\gamma+1-\alpha)\cdot\ldots\cdot(\gamma+k-1-\alpha)}{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+k-1)}\\ &\quad\times\frac{(\gamma-\beta)(\gamma+1-\beta)\cdot\ldots\cdot(\gamma+k-1-\beta)}{(\gamma-\alpha-\beta)(\gamma+1-\alpha-\beta)\cdot\ldots\cdot(\gamma+k-1-\alpha-\beta)}. \end{aligned}$$

It means that the equation (4) can be transformed to the form

$$F(\alpha,\beta,\gamma,1)=\frac{\prod(k,\gamma-1)\cdot\prod(k,\gamma-\alpha-\beta-1)}{\prod(k,\gamma-\alpha-1)\cdot\prod(k,\gamma-\beta-1)}\cdot F(\alpha,\beta,\gamma+k,1).$$

It is easily seen that $\prod(k,z)$ is defined for all $z \in \mathbb{C}$ except the negative integers. If z is a nonnegative integer, we get (for all $k \in \mathbb{N}$)

$$\begin{aligned}
\prod(k,0) &= 1,\\
\prod(k,1) &= \frac{1\cdot k}{k+1} = \frac{1}{1+\frac{1}{k}},\\
\prod(k,2) &= \frac{1\cdot 2\cdot k^2}{(k+1)(k+2)} = \frac{1\cdot 2}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)},\\
&\vdots\\
\prod(k,z) &= \frac{1\cdot 2\cdot 3\cdot\ldots\cdot z}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)\cdot\ldots\cdot\left(1+\frac{z}{k}\right)}.
\end{aligned}$$

Gauss determined the values of the function $\prod(k,z+1)$ for given k and z by the recurrence formula

$$\prod(k,z+1) = \prod(k,z)\cdot\frac{z+1}{1+\frac{z+1}{k}}.$$

In a similar way, Gauss found a recurrence formula with respect to k

$$\prod(k+1,z) = \prod(k,z)\cdot\frac{\left(1+\frac{1}{k}\right)^{z+1}}{1+\frac{1+z}{k}}. \tag{5}$$

The formula (5) yields the following equalities

$$\begin{aligned}
\prod(1,z) &= \frac{1}{1+z},\\
\prod(2,z) &= \frac{1}{1+z}\cdot\frac{\left(\frac{2}{1}\right)^{z+1}}{\frac{2+z}{1}} = \frac{1}{1+z}\cdot\frac{2^{z+1}}{2+z},\\
&\vdots\\
\prod(k,z) &= \frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z\cdot(2+z)}\cdot\frac{3^{z+1}}{2^z\cdot(3+z)}\cdot\ldots\cdot\frac{k^{z+1}}{(k-1)^z\cdot(k+z)}.
\end{aligned}$$

Then Gauss supposed $k\to\infty$, z as fixed point and set $k=h$, $z<h$. If h increase to $h+1$, then the value of $\log\prod(k,z)$ increase too. It holds by (5)

$$\log\prod(h+1,z) - \log\prod(h,z) = \log\frac{\left(1+\frac{1}{h}\right)^{z+1}}{1+\frac{1+z}{h}} = \log\frac{\left(1+\frac{1}{h}\right)^{z}}{1+\frac{z}{h+1}}. \tag{6}$$

According to the equality $\frac{h+1}{h} = \frac{1}{\frac{h}{h+1}}$ and the equation (6), Gauss got

$$\log\prod(h+1,z) - \log\prod(h,z) = -z\log\left(1-\frac{1}{h+1}\right) - \log\left(1+\frac{z}{h+1}\right). \tag{7}$$

Both logarithms on the right-hand side of (7) can be expressed in the form of power series

$$z\left(\frac{1}{h+1}+\frac{1}{2(h+1)^2}+\dots\right)-\left(\frac{z}{h+1}-\frac{z^2}{2(h+1)^2}+\dots\right).$$

Assuming the absolute convergence of both series, Gauss got for the increase of $\log\prod(h,z)$ the convergent series

$$\frac{z(1+z)}{2(h+1)^2}+\frac{z(1-z^2)}{3(h+1)^3}+\frac{z(1+z^3)}{4(h+1)^4}+\frac{z(1-z^4)}{5(h+1)^5}+\dots\,.$$

If the value k increases from $h+1$ to $h+2$, then one has

$$\log\prod(h+2,z)-\log\prod(h+1,z)=\frac{z(1+z)}{2(h+2)^2}+\frac{z(1-z^2)}{3(h+2)^3}+\frac{z(1+z^3)}{4(h+2)^4}+\dots\,.$$

Generally, if the value k increases from h to $h+n$, then the difference $\log\prod(h+n,z)-\log\prod(h,z)$ equals

$$\begin{aligned}&\frac{1}{2}z(1+z)\left(\frac{1}{(h+1)^2}+\frac{1}{(h+2)^2}+\frac{1}{(h+3)^2}+\dots+\frac{1}{(h+n)^2}\right)\\&\quad+\frac{1}{3}z(1-z^2)\left(\frac{1}{(h+1)^3}+\frac{1}{(h+2)^3}+\frac{1}{(h+3)^3}+\dots+\frac{1}{(h+n)^3}\right)\\&\quad+\frac{1}{4}z(1+z^3)\left(\frac{1}{(h+1)^4}+\frac{1}{(h+2)^4}+\frac{1}{(h+3)^4}+\dots+\frac{1}{(h+n)^4}\right)+\dots\,.\end{aligned}$$

For $n\to\infty$ Gauss obtained an absolute convergent double series

$$\sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\left(\frac{z+z^{2i}}{2i(h+j)^{2i}}+\frac{z-z^{2i+1}}{(2i+1)(h+j)^{2i+1}}\right)\right].$$

Gauss proved the finitness of $\lim_{k\to\infty}\prod(k,z)$ for every $\mathbb{C}\backslash\{-1,-2,-3,\dots\}$. Value of this limit depends on z only, hence the function $\lim_{k\to\infty}\prod(k,z)$ depends also only on z. Thus Gauss introduced the function $\prod z$ by equation

$$\prod z=\lim_{k\to\infty}\frac{1\cdot 2\cdot 3\cdot\ldots\cdot k\cdot k^z}{(z+1)(z+2)(z+3)\cdot\ldots\cdot(z+k)},$$

or by an infinite product

$$\prod z=\frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z(2+z)}\cdot\frac{3^{z+1}}{2^z(3+z)}\cdot\frac{4^{z+1}}{3^z(4+z)}\cdot\dots,$$

which represents an analog of the Euler's definition, see [7]. But there is an important difference in the domain of definition in comparison to Euler's definition. Euler considered reals without the negative integers, while Gauss took the set $\mathbb{C}\backslash\{-1,-2,-3,\dots\}$ as the domain of the function $\prod z$.

This definition of Gauss played an important role in later development of the theory of functions. For example, it was the impuls which led Karl Weierstrass to the idea about elementary factors used in his factorization theorem. The notation $\prod z$ comes from Gauss. Later, in 1809, Adrien-Marie Legendre (1752-1833) introduced a standard notation $\Gamma(z)$ instead of $\prod(z-1)$, see [6].

References

[1] C.F. Gauss, *Theoria residuorum biquadraticorum, Commentatio secunda*, in: Carl Friedrich Gauss, Werke, zweiter Band, Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1863, 95-148.

[2] C.F. Gauss, *Disquisitiones Generales Circa seriem infinitam* $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+$ *etc.*, in: Carl Friedrich Gauss, Werke, dritter Band, Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1866, 123-162.

[3] *Carl Friedrich Gauss, Werke, zehnten Dandes, erste Abteilung* , Königlichen Gesellschaft der Wissenschaften zu Göttingen, B.G. Teubner, Leipzig, 1917, 362-365.

[4] C. Kramp, *Analyse des réfractions astronomiques et terrestres*, Jacques Dannbach, Strasbourg, 1799.

[5] C. Kramp, *Élémens d'arithmétique universelle*, Cologne, 1808.

[6] A.-M. Legendre, *Memoires de la classe des sciences mathematiques et physiques de l'Institut de France*, Paris 1809, 477-490.

[7] O. Moc, *Teaching calculus with original historical sources* – Γ *function*, Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie, Matematyka **9** (2003) 165-171.

[8] K. Weierstrass, *Über die Theorie der analytischen Facultäten*, in: Mathematische Werke von Karl Weierstrass, Herausgegeben unter Mitwirkung einer von der Königlich preussischen Akademie der Wissenschaften Eingesetzten Commission, Deutsche Akademie der Wissenschaften zu Berlin. Mayer & Müller, Berlin, 1894-1927, 153-221.

Department of Mathematics and Statistics
Faculty of Social and Economic Studies
J.E. Purkyný University
Moskevská 54
Ústí nad Labem
400 96 Czech Repulic
E-mail: moc@fse.ujep.cz

Received: 20 November 2006; final version: 12 October 2007;
available online: 29 January 2008.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

V.M. Fedorchuk, V.I. Fedorchuk

On functional bases of the first-order differential invariants for non-conjugate subgroups of the Poincaré group $P(1,4)$

Abstract. It is established which functional bases of the first-order differential invariants of the splitting and non-splitting subgroups of the Poincaré group $P(1,4)$ are invariant under the subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. The obtained sets of functional bases are classified according to dimensions.

1. Introduction

It is well known (see, for example, [10, 11, 12, 13]), that functional bases of differential invariants of Lie groups of the point transformations play an important role in group analysis of differential equations, theorethical and mathematical physics, geometry, etc.

The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. Some applications of this group in the theoretical and mathematical physics can be found in [7, 8, 9].

Continuous subgroups of the group $P(1,4)$ have been described in [3, 4, 6]. One of important consequences of the study of the non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ (group symmetry of relativistic physics) and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (group symmetry of non-relativistic physics) (see also [7]).

Recently the functional bases of the first-order differential invariants for all continuous subgroups of the group $P(1,4)$ have been constructed. Some of them can be found in [2, 1].

The present paper is devoted to the classification of the functional bases of the first-order differential invariants of continuous subgroups of the group $P(1,4)$. It is established which functional bases of the first-order differential

AMS (2000) Subject Classification: 17B05, 17B81.

invariants of the splitting and non-splitting subgroups of the group $P(1,4)$ are invariant under the subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. The obtained sets of functional bases are classified according to dimensions.

In order to present some of obtained results, we consider the Lie algebra of the group $P(1,4)$.

2. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\begin{aligned} \left[P'_\mu, P'_\nu\right] &= 0, \\ \left[M'_{\mu\nu}, P'_\sigma\right] &= g_{\mu\sigma} P'_\nu - g_{\nu\sigma} P'_\mu, \\ \left[M'_{\mu\nu}, M'_{\rho\sigma}\right] &= g_{\mu\rho} M'_{\nu\sigma} + g_{\nu\sigma} M'_{\mu\rho} - g_{\nu\rho} M'_{\mu\sigma} - g_{\mu\sigma} M'_{\nu\rho}, \end{aligned}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

All non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ are divided into splitting and non-splitting ones.

Splitting subalgebras $P_{i,a}$ of the Lie algebra of the group $P(1,4)$ can be written in the following form:

$$P_{i,a} = F_i \overset{\circ}{+} N_{ia},$$

where F_i are subalgebras of the Lie algebra of the group $O(1,4)$, N_{ia} are subalgebras of the Lie algebra of the translations group $T(5) \subset P(1,4)$ and $\overset{\circ}{+}$ is the semi-direct sum.

Non-splitting subalgebras $\widetilde{P}_{j,k}$ are subalgebras, for which a basis can be chosen in the form:

$$\widetilde{B}_k = B_k + \sum_i c_{ki} X_i, \qquad \sum_j d_{rj} X_j,$$

where c_{ki} and d_{rj} are fixed real constants (not equal zero simultaneously). B_k are bases of subalgebras of the Lie algebra of the group $O(1,4)$, X_i are bases of subalgebras of the Lie algebra of the group $T(5)$.

We consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3}, \quad P'_4 = -\frac{\partial}{\partial x_4},$$

$$M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Further, we will use the following basis elements:

$$
\begin{aligned}
&G = M'_{40}\,,\\
&L_1 = M'_{32}\,, \qquad L_2 = -M'_{31}\,, \qquad L_3 = M'_{21}\,,\\
&P_a = M'_{4a} - M'_{a0}\,, \quad (a = 1,2,3),\\
&C_a = M'_{4a} + M'_{a0}\,, \quad (a = 1,2,3),\\
&X_0 = \frac{1}{2}(P'_0 - P'_4), \qquad X_k = P'_k \quad (k = 1,2,3), \qquad X_4 = \frac{1}{2}(P'_0 + P'_4).
\end{aligned}
$$

The Lie algebra of the group $\widetilde{G}(1,3)$ is generated by the following basis elements:

$$L_1\,,\ L_2\,,\ L_3\,,\ P_1\,,\ P_2\,,\ P_3\,,\ X_0\,,\ X_1\,,\ X_2\,,\ X_3\,,\ X_4\,.$$

3. The first-order differential invariants of splitting subgroups of the group $P(1,4)$

The first-order differential invariants J of any non-conjugate k-parametrical subgroup of the group $P(1,4)$ can be obtained as solutions of the following systems of differential equations:

$$
\begin{cases}
\widetilde{X}_1 J(x_0,x_1,x_2,x_3,x_4,u,u_0,u_1,u_2,u_3,u_4) = 0,\\
\widetilde{X}_2 J(x_0,x_1,x_2,x_3,x_4,u,u_0,u_1,u_2,u_3,u_4) = 0,\\
\qquad\vdots\\
\widetilde{X}_k J(x_0,x_1,x_2,x_3,x_4,u,u_0,u_1,u_2,u_3,u_4) = 0,
\end{cases}
$$

where $\{\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_k,\ (k = 1,\ldots,12,15)\}$ are one times prolonged basis operators of any k-dimensional subalgebras of the Lie algebra of group $P(1,4)$, u is an arbitrary smooth function on $M(1,4)$, $u_\mu \equiv \frac{\partial u}{\partial x_\mu}$, $\mu = 0,1,2,3,4$.

Any solution of this system can be written in the following form:

$$J(x_0,x_1,x_2,x_3,x_4,u,u_0,u_1,u_2,u_3,u_4) = F(J_1, J_2, \ldots, J_t),$$

where $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of the considered subalgebra, F is an arbitrary smooth function. In this formula

$$J_i = J_i(x_0,x_1,x_2,x_3,x_4,u,u_0,u_1,u_2,u_3,u_4), \qquad i = 1,\ldots,t.$$

More details about solutions construction of the above mentioned type systems as well as the solved examples can be found in [10, 12, 13].

Using the prolongation theory (see, for example, [12, 13]) we have constructed the first prolongation for basis operators of the Lie algebra of the group $P(1,4)$. One times prolonged bases operators of the Lie algebra of the group $P(1,4)$ have the following form:

$$\widetilde{G} = -x_4\frac{\partial}{\partial x_0} - x_0\frac{\partial}{\partial x_4} + u_4\frac{\partial}{\partial u_0} + u_0\frac{\partial}{\partial u_4},$$

$$\widetilde{L}_1 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} + u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3},$$

$$\widetilde{L}_2 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3} - u_3\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_3},$$

$$\widetilde{L}_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2},$$

$$\widetilde{P}_1 = x_1\frac{\partial}{\partial x_0} + (x_0+x_4)\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_4} - u_1\frac{\partial}{\partial u_0} - (u_0-u_4)\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_4},$$

$$\widetilde{P}_2 = x_2\frac{\partial}{\partial x_0} + (x_0+x_4)\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_4} - u_2\frac{\partial}{\partial u_0} - (u_0-u_4)\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_4},$$

$$\widetilde{P}_3 = x_3\frac{\partial}{\partial x_0} + (x_0+x_4)\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4} - u_3\frac{\partial}{\partial u_0} - (u_0-u_4)\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4},$$

$$\widetilde{C}_1 = -x_1\frac{\partial}{\partial x_0} - (x_0-x_4)\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_4} + u_1\frac{\partial}{\partial u_0} + (u_0+u_4)\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_4},$$

$$\widetilde{C}_2 = -x_2\frac{\partial}{\partial x_0} - (x_0-x_4)\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_4} + u_2\frac{\partial}{\partial u_0} + (u_0+u_4)\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_4},$$

$$\widetilde{C}_3 = -x_3\frac{\partial}{\partial x_0} - (x_0-x_4)\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4} + u_3\frac{\partial}{\partial u_0} + (u_0+u_4)\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4},$$

$$\widetilde{X}_0 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_4}\right), \qquad \widetilde{X}_1 = -\frac{\partial}{\partial x_1}, \qquad \widetilde{X}_2 = -\frac{\partial}{\partial x_2},$$

$$\widetilde{X}_3 = -\frac{\partial}{\partial x_3}, \qquad \widetilde{X}_4 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_4}\right).$$

In the mentioned above denotations this basis can be written as $\{\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_{15}\}$.

Going through the list of all non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ presented in [5] one derives that the set of functional bases of the first-order differential invariants of the splitting subgroups of the group $P(1,4)$ contains 99 ones which are invariant under the splitting subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. It is impossible to present all these bases here. Therefore, below we give only a short review of the results obtained. In each example we write the basis elements of the splitting subalgebras of the Lie algebra of the group $\widetilde{G}(1,3)$ and their functional basis.

1. There is 1 three-dimensional functional basis

$$\langle X_1 \equiv P_1\,,\ X_2 \equiv P_2\,,\ X_3 \equiv P_3\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_3 - P_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv P_3\,,\ X_5 \equiv X_0\,,\ X_6 \equiv X_1\,,\ X_7 \equiv X_2\,,\ X_8 \equiv X_3\,,\ X_9 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_1\,,\ X_2 \equiv L_2\,,\ X_3 \equiv L_3\,,\ X_4 \equiv P_1\,,\ X_5 \equiv P_2\,,\ X_6 \equiv P_3\,,\ X_7 \equiv X_0\,,\ X_8 \equiv X_1\,,\ X_9 \equiv X_2\,,\ X_{10} \equiv X_3\,,\ X_{11} \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,, \qquad J_3 = u_0 - u_4\,;$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \qquad (\mu = 0, 1, 2, 3, 4).$$

2. There are 4 four-dimensional functional bases. For example

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv P_3\,,\ X_3 \equiv X_0\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_3\,,\ X_7 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,,$$
$$J_3 = u_0 - u_4\,, \qquad J_4 = u_1^2 + u_2^2\,.$$

3. There are 12 five-dimensional functional bases. For example

$$\langle X_1 \equiv P_1\,,\ X_2 \equiv P_2\,,\ X_3 \equiv X_1\,,\ X_4 \equiv X_2\,,\ X_5 \equiv X_3\,,\ X_6 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_3\,,\ X_7 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,, \qquad J_3 = x_0 + x_4\,,$$
$$J_4 = u_3\,, \qquad J_5 = u_0 - u_4\,.$$

4. There are 19 six-dimensional functional bases. For example

$$\langle X_1 \equiv L_1\,,\ X_2 \equiv L_2\,,\ X_3 \equiv L_3\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,,$$
$$J_3 = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \qquad J_4 = x_1 u_1 + x_2 u_2 + x_3 u_3\,,$$
$$J_5 = u_0\,, \qquad J_6 = u_4\,.$$

5. There are 26 seven-dimensional functional bases. For example

$$\langle X_1 \equiv P_3\,,\ X_2 \equiv X_0\,,\ X_3 \equiv X_3\,,\ X_4 \equiv X_4\rangle,$$

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2, \quad J_3 = x_1,$$
$$J_4 = x_2, \quad J_5 = u_1, \quad J_6 = u_2,$$
$$J_7 = u_0 - u_4.$$

6. There are 20 eight-dimensional functional bases. For example

$\langle X_1 \equiv L_3, \ X_2 \equiv X_0, \ X_3 \equiv X_4 \rangle$,

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2, \quad J_3 = x_3,$$
$$J_4 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad J_5 = x_1 u_2 - x_2 u_1, \quad J_6 = u_0,$$
$$J_7 = u_3, \quad J_8 = u_4.$$

7. There are 11 nine-dimensional functional bases. For example

$\langle X_1 \equiv L_3 - P_3, \ X_2 \equiv X_4 \rangle$,

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2,$$
$$J_3 = x_0 + x_4, \quad J_4 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$
$$J_5 = \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + x_4}, \quad J_6 = x_1 u_2 - x_2 u_1,$$
$$J_7 = \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4}, \quad J_8 = u_0 - u_4,$$
$$J_9 = u_1^2 + u_2^2.$$

8. There are 6 ten-dimensional functional bases. For example

$\langle X_1 \equiv P_3 \rangle$,

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2,$$
$$J_3 = x_1, \quad J_4 = x_2,$$
$$J_5 = x_0 + x_4, \quad J_6 = (x_0^2 - x_3^2 - x_4^2)^{\frac{1}{2}},$$
$$J_7 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_8 = u_0 - u_4,$$
$$J_9 = u_1, \quad J_{10} = u_2.$$

4. The first-order differential invariants of the non-splitting subgroups of the group $P(1,4)$

As in the Section 3, it is established that the set of functional bases of the first-order differential invariants of the non-splitting subgroups of the group $P(1,4)$ contains 158 ones which are invariant under the non-splitting subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. It is impossible to present all these bases here. Therefore, below we give only a short review of the results obtained. In each example we write the basis elements of the non-splitting subalgebras of the Lie algebra of the group $\widetilde{G}(1,3)$ and their functional basis.

1. There is 1 three-dimensional functional basis

$$\langle X_1 \equiv L_3 - X_0\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv P_3\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,$$

$$\langle X_1 \equiv P_1\,,\ X_2 \equiv P_2\,,\ X_3 \equiv P_3 + X_0\,,\ X_4 \equiv L_3 + \beta X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,\ \beta < 0,$$

$$J_1 = u, \qquad J_2 = u_0 - u_4\,, \qquad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,;$$
$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad (\mu = 0,1,2,3,4).$$

2. There are 5 four-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 + d_3 X_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_4\rangle,\ d_3 < 0,$$

$$\langle X_1 \equiv L_3 - X_0\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_3\,,\ X_7 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_3\,, \qquad J_3 = u_0 - u_4\,, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_4^2\,.$$

3. There are 12 five-dimensional functional bases. For example

$$\langle X_1 \equiv P_1 + \beta X_3\,,\ X_2 \equiv P_2\,,\ X_3 \equiv P_3 + X_0\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_4\rangle,\ \beta > 0,$$

$$J_1 = u, \qquad J_2 = (x_0 + x_4) + \frac{u_3}{u_0 - u_4}\,,$$
$$J_3 = (x_0 + x_4)^2 - 2x_3 + 2\beta\frac{u_1}{u_0 - u_4}\,, \qquad J_4 = u_0 - u_4\,,$$
$$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,.$$

4. There are 34 six-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 + dX_3\,,\ X_2 \equiv P_3\,,\ X_3 \equiv X_1\,,\ X_4 \equiv X_2\,,\ X_5 \equiv X_4\rangle,\ d < 0,$$

$$J_1 = x_0 + x_4\,, \qquad J_2 = u, \qquad J_3 = x_3 + d\arctan\frac{u_1}{u_2} + u_3\frac{x_0 + x_4}{u_0 - u_4},$$
$$J_4 = u_0 - u_4\,, \qquad J_5 = u_1^2 + u_2^2\,, \quad J_6 = u_0^2 - u_3^2 - u_4^2\,.$$

5. There are 49 seven-dimensional functional bases. For example

$$\langle X_1 \equiv P_1 + \delta X_3\,,\ X_2 \equiv P_2 + X_3\,,\ X_3 \equiv X_1,\ X_4 \equiv X_4\rangle,\ \delta > 0,$$

$$J_1 = x_0 + x_4\,, \qquad J_2 = u, \qquad J_3 = \frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4},$$
$$J_4 = \frac{\delta u_1 + u_2}{u_0 - u_4} - x_3\,, \qquad J_5 = u_3\,, \qquad J_6 = u_0 - u_4\,,$$
$$J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2\,.$$

6. There are 32 eight-dimensional functional bases. For example

$$\langle X_1 \equiv P_3 + X_0\,,\ X_2 \equiv X_1\,,\ X_3 \equiv X_4\rangle,$$

$$J_1 = x_2\,, \qquad J_2 = (x_0 + x_4)^2 - 2x_3\,, \qquad J_3 = u,$$
$$J_4 = (x_0 + x_4) + \frac{u_3}{u_0 - u_4}, \qquad J_5 = u_1\,, \qquad J_6 = u_2\,,$$
$$J_7 = u_0 - u_4\,, \qquad J_8 = u_0^2 - u_3^2 - u_4^2\,.$$

7. There are 19 nine-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 - X_4\,,\ X_2 \equiv X_3\rangle,$$

$$J_1 = x_0 + x_4\,, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_3 = u,$$
$$J_4 = x_1 u_2 - x_2 u_1\,, \qquad J_5 = \arctan\frac{u_1}{u_2} + x_0 - x_4\,, \qquad J_6 = u_0\,,$$
$$J_7 = u_3\,, \qquad J_8 = u_4\,, \qquad J_9 = u_1^2 + u_2^2\,.$$

8. There are 6 ten-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 - P_3 + \alpha_0 X_0\rangle,\ \alpha_0 < 0,$$

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_2 = (x_0 + x_4)^2 + 2\alpha_0 x_3\,,$$
$$J_3 = x_1 u_2 - x_2 u_1\,, \qquad J_4 = \alpha_0 \arctan\frac{x_1}{x_2} - x_0 - x_4\,,$$
$$J_5 = x_0 + x_4 - \alpha_0 \frac{u_3}{u_0 - u_4}, \qquad J_6 = 2(x_0 + x_4)^3 + 6\alpha_0 x_3 (x_0 + x_4) + 3\alpha_0^2 (x_0 - x_4),$$
$$J_7 = u, \qquad J_8 = u_0 - u_4\,,$$
$$J_9 = u_1^2 + u_2^2\,, \qquad J_{10} = u_0^2 - u_3^2 - u_4^2\,.$$

As we see there are not the functional bases with dimensions less than 3 as well as ones with dimensions bigger than 10. It follows from using of the theorem on invariants of Lie groups of the point transformations for all non-conjugate subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. More details about this theorem can be found in [12, 13].

The results obtained can be used for the construction and investigation of classes of first–order differential equations (defined in the space $M(1,4) \times R(u)$)

invariant under continuous subgroups of the group $\widetilde{G}(1,3) \subset P(1,4)$. $R(u)$ is the axis of the dependent variable u.

References

[1] V.M. Fedorchuk, V.I. Fedorchuk, *On the differential first-order invariants for the non-splitting subgroups of the generalized Poincaré group $P(1,4)$*, Ann. Acad. Pedagog. Crac. Stud. Math. **4** (2004), 65-74.

[2] V.M. Fedorchuk, V.I. Fedorchuk, *On differential invariants of first- and second-order of the splitting subgroups of the generalized Poincaré group $P(1,4)$*, Symmetry in nonlinear mathematical physics, Part 1, 2 (Kyiv, 2001), Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., **43**, Part 1, vol. 2, Natsīonal. Akad. Nauk Ukraïni Īnst. Mat., Kiev, 2002, 140-144.

[3] V.M. Fedorchuk, *Nonsplit subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$*, Ukrain. Mat. Zh. **33** (1981), no. 5, 696-700.

[4] V.M. Fedorčuk, *Split subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$*, Ukrain. Mat. Zh. **31** (1979), no. 6, 717-722.

[5] V.I. Fushchich, L.F. Barannik, A.F. Barannik, *Podgruppovoi analiz grupp Galileya, Puankare i reduktsiya nelineinykh uravnenii*, "Naukova Dumka", Kiev, 1991.

[6] W.I. Fushchich, A. F. Barannik, L. F. Barannik, V. M. Fedorchuk, *Continuous subgroups of the Poincaré group $P(1,4)$*, J. Phys. A: Math. Gen **18** (1985), no. 15, 2893-2899.

[7] W.I. Fushchich, A.G. Nikitin, *Symmetries of equations of quantum mechanics*, Allerton Press Inc., New York, 1994.

[8] W.I. Fushchych, *Representations of the complete inhomogeneous de Sitter group and equations in the five-dimensional approach. I*, Theor. Math. Phys. **4** (1970), no. 3, 890-907.

[9] V.G. Kadyševskiĭ, *A new approach to the theory of electromagnetic interactions*, Fiz. Èlementar. Chastits i Atom. Yadra **11** (1980), no. 1, 5–39.

[10] S. Lie, *Ueber Differentialinvarianten*, Math. Ann. **24** (1884), no. 4, 537-578.

[11] S. Lie, G. Scheffers, *Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen*, Teubner, Leipzig, 1893.

[12] P.J. Olver, *Applications of Lie groups to differential equations*, Graduate Texts in Mathematics, vol. **107**, Springer-Verlag, New York, 1986.

[13] L.V. Ovsiannikov, *Group analysis of differential equations*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982.

Institute of Mathematics
Pedagogical University,
Podchorążych 2
30-084, Kraków
Poland;
Pidstryhach Institute of Applied Problems
of Mechanics and Mathematics
of the National Ukrainian Academy of Sciences
3b Naukova Street
Lviv, 79601
Ukraine E-mail: fedorchuk@ap.krakow.pl

Pidstryhach Institute of Applied Problems
of Mechanics and Mathematics
of the National Ukrainian Academy of Sciences
3b Naukova Street
Lviv, 79601
Ukraine
E-mail: fedorchuk@gmail.com

Received: 20 November 2006; final version: 5 December 2007;
available online: 12 May 2008.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Marek Czerni

Another description of continuous solutions of a nonlinear functional inequality

Abstract. The paper gives a general construction of all continuous solutions of inequality (1) fulfilling one of conditions (5) or (26). This paper is a continuation of [3].

1. Introduction

In the paper [3] we considered the problem of existence of the continuous solutions of the functional inequality

$$\psi[f(x)] \leq G(x, \psi(x)), \tag{1}$$

where ψ is an unknown function, in the case where continuous solutions of the corresponding functional equation

$$\varphi[f(x)] = G(x, \varphi(x)) \tag{2}$$

depend on an arbitrary function. In particular we proved there Theorems 1 and 5 quoted below.

In the present paper we shall give other descriptions of the general continuous solution of (1) which are more convenient to study, for example, solutions of (1) which are Lipschitzian or possess some asymptotic property (see [2], [1]). We shall also adapt some results from [3] to a more general class of continuous solutions of inequality (1).

We start with reminding some notations and assumptions from [3]. Let $I = (\xi, a)$, where $\xi < a \leq \infty$. We assume that

(i) the function $f\colon I \longrightarrow \mathbb{R}$ is continuous and strictly increasing in I. Moreover, $\xi < f(x) < x$ for all $x \in I$.

Remark 1

Hypothesis (i) implies that $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$. Here f^n denotes the n-th iterate of f.

AMS (2000) Subject Classification: 39B62.

As to the function G we assume:

(ii) $G: \Omega \longrightarrow \mathbb{R}$ is continuous in an open set $\Omega \subset I \times \mathbb{R}$;

(iii) for every $x \in I$ the set

$$\Omega_x := \{y : \ (x, y) \in \Omega\} \tag{3}$$

is a non-empty open interval and

$$G(x, \Omega_x) \subset \Omega_{f(x)}. \tag{4}$$

Let $J \subset I$ be an open subinterval such that $\xi \in \operatorname{cl} J$. We shall consider solutions ψ of inequality (1) and solutions φ of equation (2) such that their graphs lie in Ω, i.e.,

$$\psi(x), \varphi(x) \in \Omega_x \qquad \text{for } x \in J \subset I. \tag{5}$$

The class of these solutions will be denoted by $\Psi(J)$ and $\Phi(J)$, respectively. Moreover, we denote $I_k := [f^{k+1}(x_0), f^k(x_0)]$ for a fixed $x_0 \in I$ and $k \in \mathbb{N} \cup \{0\}$.

Finally, we consider the sequence $\{g_k\}$ defined by the recursive formula:

$$\begin{cases} g_0(x, y) = y, \\ g_{k+1}(x, y) = G(f^k(x), g_k(x, y)), \quad k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{6}$$

2. Solutions of (1) in the interval $(\xi, x_0]$

Let us assume (i)-(iii). It is known (see [4]) that then continuous solutions of equation (2) depend on an arbitrary function. It means that for any $x_0 \in I$ and an arbitrary continuous function $\varphi_0: I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions

$$\varphi_0(x) \in \Omega_x \qquad \text{for } x \in I_0\,, \tag{7}$$

$$\varphi_0[f(x_0)] = G(x_0, \varphi_0(x_0)) \tag{8}$$

there exists exactly one continuous solution $\varphi \in \Phi((\xi, x_0])$ of equation (2) extending φ_0, i.e.,

$$\varphi(x) = \varphi_0(x) \qquad \text{for } x \in I_0\,.$$

A corresponding result for solutions of inequality (1) has been proved in [3]:

Theorem 1
Let assumptions (i)-(iii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0: I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions*

$$\psi_0[f(x_0)] \le G(x_0, \psi_0(x_0)), \tag{9}$$

$$\psi_0(x) \in \Omega_x\,, \qquad x \in I_0 \tag{10}$$

there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *such that*

$$\psi(x) = \psi_0(x) \qquad \text{for } x \in I_0 \,. \tag{11}$$

This solution is given by the formula

$$\psi[f^k(x)] = \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \qquad \text{for } x \in I_0\,,\ k \in \mathbb{N} \cup \{0\}, \tag{12}$$

where $\lambda_k\colon I_k \longrightarrow \mathbb{R}$ *is an arbitrary sequence of continuous functions fulfilling the conditions:*

$$\lambda_0(x) = 0, \qquad x \in I_0\,, \tag{13}$$

$$\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \in \Omega_{f^k(x)}\,, \qquad x \in I_0\,,\ k \in \mathbb{N} \cup \{0\}, \tag{14}$$

$$\begin{aligned}\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \le G(f^{-k}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))),\\ x \in I_0\,,\ k \in \mathbb{N},\end{aligned} \tag{15}$$

$$\begin{aligned}\lambda_k[f^k(x_0)] + g_k(x_0, \psi_0(x_0)) = \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)]),\\ k \in \mathbb{N}.\end{aligned} \tag{16}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *may be obtained in this manner.*

Now, we shall prove the following corollary from Theorem 1.

THEOREM 2
Let assumptions (i)-(iii) *be fulfilled. Then for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ *fulfilling* (9) *and* (10) *there exists a continuous solution* $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *such that* (11) *holds. This solution is given by the formula*

$$\psi(x) = \begin{cases} \psi_0(x), & x \in I_0\,, \\ M_k(\psi_0, \lambda)(x), & x \in I_k\,,\ k \in \mathbb{N}, \end{cases} \tag{17}$$

where the functional sequence of continuous functions $M_k(\psi_0, \lambda)$ *is defined by the recurrence*

$$\begin{cases} M_1(\psi_0, \lambda)(x) = \lambda(x) + G(f^{-1}(x), \psi_0[f^{-1}(x)]), & x \in I_1\,, \\ M_{k+1}(\psi_0, \lambda)(x) = \lambda(x) + G(f^{-1}(x), M_k(\psi_0, \lambda)[f^{-1}(x)]), & x \in I_{k+1} \end{cases} \tag{18}$$

and $\lambda\colon (\xi, f(x_0)] \longrightarrow (-\infty, 0]$ *is an arbitrary continuous function fulfilling the conditions:*

$$M_k(\psi_0, \lambda)(x) \in \Omega_x\,, \qquad x \in I_k\,,\ k \in \mathbb{N}, \tag{19}$$

$$\lambda[f(x_0)] + G(x_0, \psi_0(x_0)) = \psi_0[f(x_0)]. \tag{20}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *may be obtained in this manner.*

Proof. We fix an $x_0 \in I$ and an arbitrary continuous function $\psi_0 \colon I_0 \longrightarrow \mathbb{R}$ fulfilling (9) and (10). Moreover, we take a continuous function $\lambda \colon (\xi, f(x_0)] \longrightarrow (-\infty, 0]$ fulfilling (19), (20) and define the function $\psi \colon (\xi, x_0] \longrightarrow \mathbb{R}$ by formula (17). Condition (19) implies that the sequence $M_k(\psi_0, \lambda)$ (and, consequently, the function ψ) is well defined. Now, we define the sequence $\lambda_k \colon I_k \longrightarrow \mathbb{R}$ of continuous functions by formula (13) and

$$\lambda_k(x) := \lambda(x) + G(f^{-1}(x), \psi[f^{-1}(x)]) - g_k(f^{-k}(x), \psi[f^{-k}(x)]), \qquad x \in I_k\,,\ k \in \mathbb{N}. \tag{21}$$

It is obvious that (21) implies that ψ may be represented also by formula (12). Moreover, condition (19) implies (14). We have also the estimate

$$\begin{aligned} \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) &= \lambda_k[f^k(x)] + G(f^{k-1}(x), \psi[f^{k-1}(x)]) \\ &\le G(f^{k-1}(x), \psi[f^{k-1}(x)]) \\ &= G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))), \\ &\qquad x \in I_0\,,\ k \in \mathbb{N}, \end{aligned}$$

which implies (15). Finally from (20) we obtain (16) for $k = 1$ and, by virtue of the equalities

$$\begin{aligned} &\lambda_k[f^k(x_0)] + g_k(x_0, \psi_0(x_0)) \\ &\quad = \lambda[f^k(x_0)] + G(f^{k-1}(x_0), \psi[f^{k-1}(x_0)]) \\ &\quad = \lambda[f^{k-1}(f(x_0))] + G(f^{k-2}(f(x_0)), \psi[f^{k-2}(f(x_0))]) \\ &\quad = \lambda_{k-1}[f^{k-1}(f(x_0))] + g_{k-1}(f(x_0), \psi_0[f(x_0)]) \\ &\quad = \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)]), \end{aligned}$$

we have (16) for $k \ge 2$. Thus, by virtue of Theorem 1, formula (12) (and, consequently, formula (17)) defines a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality (1).

On the other hand, let us assume that $\psi \in \Psi((\xi, x_0])$ is a continuous solution of (1). It is sufficient to put

$$\psi_0(x) := \psi(x) \qquad \text{for } x \in I_0\,, \tag{22}$$

$$\lambda(x) := \psi(x) - G(f^{-1}(x), \psi[f^{-1}(x)]) \qquad \text{for } x \in (\xi, f(x_0)]. \tag{23}$$

Let us notice that (19) and (20) hold. Moreover, it follows from (1) that the function λ takes nonpositive values only. It is obvious that the solution ψ may be represented by formula (17). We may prove it by simple induction.

Indeed, formulas (22) and (23) imply that for $x \in I_1$:

$$\begin{aligned} \psi(x) &= \lambda(x) + G(f^{-1}(x), \psi[f^{-1}(x)]) = \lambda(x) + G(f^{-1}(x), \psi_0[f^{-1}(x)]) \\ &= M_1(\psi_0, \lambda)(x). \end{aligned}$$

Thus, if we assume that for an arbitrarily chosen integer $k > 1$ we have $\psi(x) =$

$M_k(\psi_0, \lambda)(x)$ for $x \in I_k$, then from (23) we obtain for $x \in I_{k+1}$:

$$\psi(x) = \lambda(x) + G(f^{-1}(x), \psi[f^{-1}(x)]) = \lambda(x) + G(f^{-1}(x), M_k(\psi_0, \lambda)[f^{-1}(x)])$$
$$= M_{k+1}(\psi_0, \lambda)(x).$$

Consequently, ψ is of the form (17) and this ends the proof of the theorem.

3. Solutions of (1) in the interval I

We assume additionally that:

(iv) for every $x \in I$ the function $G(x, \cdot)$ is invertible,

(v) the function f fulfils the condition $f(I) = I$,

(vi) for every $x \in I$, with Ω_x defined by (3) we have

$$G(x, \Omega_x) = \Omega_{f(x)} . \tag{24}$$

Thanks to these assumptions we may extend the definition (6) to negative indices by putting

$$g_{-k-1}(x, y) := G^{-1}(f^{-k-1}(x), g_{-k}(x, y)), \qquad k \in \mathbb{N} \cup \{0\}, \tag{25}$$

where $G^{-1}(x, \cdot)$ denotes the inverse of the function $G(x, \cdot)$. It is obvious (by virtue of (4) and (24)) that the sequences (6) and (25) are well defined. We may also consider intervals I_k for $k \in \mathbb{Z}$.

If we assume (i)-(vi), then for an arbitrary $x_0 \in I$ every continuous function $\varphi_0 \colon I_0 \longrightarrow \mathbb{R}$ fulfilling (7), (8) may be extended to a continuous solution $\varphi \in \Phi(I)$ of equation (2). For inequality (1) the following theorem has been also formulated in [3].

Theorem 3
Let assumptions (i)-(vi) *be fulfilled. Then, for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 \colon I_0 \longrightarrow \mathbb{R}$ fulfilling* (9) *and* (10) *there exists a continuous solution $\psi \in \Psi(I)$ of inequality* (1) *such that* (11) *holds. This solution is given by formulas* (12) *and*

$$\psi[f^{-k}(x)] = l_k[f^{-k}(x)] + g_{-k}(x, \psi_0(x)) \qquad \text{for } x \in I_0 , \ k \in \mathbb{N},$$

where $\lambda_k \colon I_k \longrightarrow \mathbb{R}$, $l_k \colon I_{-k} \longrightarrow \mathbb{R}$ are arbitrary sequences of continuous functions fulfilling conditions (13)-(16) *and, additionally, the following conditions*

$$l_0(x) = 0, \qquad x \in I_0 ,$$

$$l_k[f^{-k}(x)] + g_{-k}(x, \psi_0(x)) \in \Omega_{f^{-k}(x)} , \qquad x \in I_0 , \ k \in \mathbb{N},$$

$$l_{k+1}[f^{-k+1}(x)] + g_{-k+1}(x,\psi_0(x)) \le G(f^{-k}(x), l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x))),$$
$$x \in I_0\,,\ k \in \mathbb{N},$$

$$l_{k+1}[f^{-k+1}(x_0)] + g_{-k+1}(x_0,\psi_0(x_0)) = l_k[f^{-k+1}(x_0)] + g_{-k}(f(x_0),\psi_0[f(x_0)]),$$
$$k \in \mathbb{N}.$$

Moreover, we may obtain in this way all continuous solutions $\psi \in \Psi(I)$ *of inequality* (1)*.*

Theorems 2 and 3 also imply the following theorem.

THEOREM 4
Under assumptions (i)-(vi) *for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ *fulfilling the conditions* (9), (10)*, there exists a continuous solution* $\psi \in \Psi(I)$ *of inequality* (1) *such that* (11) *holds. This solution is given by formulas* (17) *and*

$$\psi(x) := P_k(\psi_0,\lambda)(x), \qquad x \in I_{-k}\,,\ k \in \mathbb{N},$$

where the functional sequences of continuous functions $M_k(\psi_0,\lambda)$*,* $P_k(\psi_0,\lambda)$*, are defined by formula* (18) *and by*

$$\begin{cases} P_1(\psi_0,\lambda)(x) = G^{-1}(x,\psi_0[f(x)] - \lambda[f(x)]), & x \in I_{-1}\,, \\ P_{k+1}(\psi_0,\lambda)(x) = G^{-1}(x,P_k(\psi_0,\lambda)[f(x)]), & x \in I_{-k-1}\,,\ k \in \mathbb{N} \end{cases}$$

and $\lambda\colon I \longrightarrow (-\infty,0]$ *is an arbitrarily chosen continuous function fulfilling conditions* (19), (20) *together with*

$$\psi_0(x) - \lambda(x) \in \Omega_x\,, \qquad x \in I_0\,,$$
$$P_k(\psi_0,\lambda)(x) \in \Omega_x\,, \qquad x \in I_{-k}\,,\ k \in \mathbb{N}.$$

Moreover, all continuous solutions $\psi \in \Psi(I)$ *of inequality* (1) *may be obtained in this manner.*

The proof of the above theorem runs analogously to that of Theorem 2 and is therefore omitted.

4. Main result

Here we shall characterize continuous solutions ψ of inequality (1) which fulfil, for arbitrarily chosen $x_0 \in I$, the additional condition

$$\psi[f(x)] \in G(x,\Omega_x), \qquad x \in (\xi, x_0]. \tag{26}$$

We replace (iv) by a stronger assumption

(vii) For every $x \in I$ the function $G(x,\cdot)$ is strictly increasing.

In the paper [3] we considered continuous solutions ψ of (1) which fulfil the following condition:

$$L_k^{\psi}[f(x)] \in G(x, \Omega_x), \qquad x \in (\xi, x_0],\ x_0 \in I,\ k \in \mathbb{N} \cup \{0\}, \tag{27}$$

where the sequence $\{L_k^{\psi}\}$ was defined by the recurrence

$$\begin{cases} L_0^{\psi}(x) = \psi(x), \\ L_{k+1}^{\psi}(x) = G^{-1}(x, L_k^{\psi}[f(x)]), \qquad k \in \mathbb{N} \cup \{0\}. \end{cases}$$

It is obvious, by virtue of (27), that the above sequence is well defined.

The following theorem has been proved in [3].

Theorem 5
Let assumptions (i)-(iii) *and* (vii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ fulfilling* (9), (10) *and, moreover, the condition*

$$\psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}) \tag{28}$$

there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality (1) *fulfilling* (11) *and* (27). *This solution is given by the formula*

$$\psi[f^k(x)] = g_k(x, \gamma_k(x) + \psi_0(x)) \qquad \text{for } x \in I_0\,,\ k \in \mathbb{N} \cup \{0\}, \tag{29}$$

where $\{\gamma_k\}$ is an arbitrary sequence of continuous functions defined in I_0 and fulfilling the conditions:

$$\gamma_0(x) = 0, \qquad x \in I_0\,,$$

$$\{\gamma_k\} \text{ is decreasing in } I_0\,,$$

$$\gamma_k(x) + \psi_0(x) \in \Omega_x\,, \qquad \text{for } x \in (f(x_0), x_0],\ k \in \mathbb{N} \cup \{0\},$$

$$\gamma_k[f(x_0)] + \psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}), \qquad k \in \mathbb{N},$$

$$g_k(x_0, \gamma_k(x_0) + \psi_0(x_0)) = g_{k-1}(f(x_0), \gamma_{k-1}[f(x_0)] + \psi_0[f(x_0)]), \qquad k \in \mathbb{N}.$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ of inequality (1), *fulfilling* (27) *may be obtained in this manner.*

If we replace in the above theorem condition (27) by (26) then we obtain:

Theorem 6
Let assumptions (i)-(iii) *and* (vii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ fulfilling* (9), (10), (28), *there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality* (1) *fulfilling* (11) *and* (26). *This solution is given by the formula*

$$\psi(x) = \begin{cases} \psi_0(x), & x \in I_0\,, \\ R_k(\psi_0,\gamma)(x), & x \in I_k\,,\ k \in \mathbb{N}, \end{cases} \tag{30}$$

where the sequence $\{R_k(\psi_0,\gamma)\}$ *of continuous functions is defined recursively by*

$$\begin{cases} R_1(\psi_0,\gamma)(x) = G(f^{-1}(x), \gamma[f^{-1}(x)] + \psi_0[f^{-1}(x)]), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in I_1\,, \\ R_{k+1}(\psi_0,\gamma)(x) = G(f^{-1}(x), \gamma[f^{-1}(x)] + R_k(\psi_0,\gamma)[f^{-1}(x)]), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in I_{k+1}\,,\ k \in \mathbb{N}, \end{cases} \tag{31}$$

and $\gamma\colon (\xi, x_0] \longrightarrow (-\infty, 0]$ *is an arbitrary continuous function fulfilling the conditions:*

$$\gamma(x) + R_k(\psi_0,\gamma)(x) \in \Omega_x\,, \qquad x \in I_k\,,\ k \in \mathbb{N} \cup \{0\}, \tag{32}$$

$$\psi_0[f(x_0)] = G(x_0, \gamma(x_0) + \psi_0(x_0)). \tag{33}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *fulfilling* (26) *may be obtained in this manner.*

Proof. Similarly as in the proof of Theorem 2 we fix an $x_0 \in I$ and an arbitrary continuous function $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ fulfilling (9), (10) and (28). Moreover, let $\gamma\colon (\xi, x_0] \longrightarrow (-\infty, 0]$ be a continuous function fulfilling (32), (33) and define a function $\psi\colon (\xi, x_0] \longrightarrow \mathbb{R}$ by formula (30). Condition (32) implies that the sequence $\{R_k(\psi_0,\gamma)\}$ is well defined. It is also clear that the following equalities

$$R_{k+1}(\psi_0,\gamma)[f^{k+1}(x_0)] = R_k(\psi_0,\gamma)[f^k(x_0)], \qquad k \in \mathbb{N}, \tag{34}$$

hold. Thus (34) together with (33) imply that the function ψ is well defined. Since $R_k(\psi_0,\gamma)$ are continuous functions (by the continuity of the given functions f, γ, ψ_0, G), so is ψ.

It is obvious that ψ may be represented by the following form, equivalent to (30),

$$\psi[f(x)] = G(x, \gamma(x) + \psi(x)), \qquad x \in (\xi, x_0]. \tag{35}$$

Equality (35) implies condition (26) and, moreover, we obtain that ψ fulfils inequality (1) by virtue of (vii) and the fact that γ takes nonpositive values only.

On the other hand let us assume that $\psi \in \Psi((\xi, x_0])$ is a continuous solution of (1) that fulfils (26). It is sufficient to define ψ_0 by (22) and to put

$$\gamma(x) := G^{-1}(x, \psi[f(x)]) - \psi(x), \qquad x \in (\xi, x_0]. \tag{36}$$

Let us notice that (9), (10), (28), (32) and (33) hold. It is obvious that the solution ψ may be represented by (35) and, consequently, by (30). This ends the proof of the theorem.

Remark 2
If we are confined to solutions ψ of inequality (1) fulfilling (27), then formulas (29) and (30) are equivalent. Indeed, if we define solution ψ fulfilling (27) by formula (30) then we may define the sequence $\{\gamma_k\}$ from Theorem 5 by the formula

$$\gamma_k(x) = L_k^{\psi}(x) - \psi_0(x), \qquad x \in I_0\,,\ k \in \mathbb{N}.$$

Conversely, if we define a solution ψ by formula (29), then we may define the function γ by (36) and the functional sequence $\{R_k(\psi_0, \gamma)\}$ of continuous functions by the recurrent formula (31).

Remark 3
It is known (see [3]) that contrary to the situation with continuous solutions of equation (2) in I, a continuous function ψ_0 fulfilling (9), (10) and (28) cannot be extended uniquely to a continuous solution ψ of inequality (1) fulfilling (26).

References

[1] M. Czerni, *Solutions of a functional inequality in a special class of functions*, in: Functional equations and inequalities, Math. Appl., vol. **518**, Kluwer Acad. Publ., Dordrecht, 2000, 55-63.

[2] M. Czerni, *On dependence of Lipschitzian solutions of nonlinear functional inequality on an arbitrary function*, in: Functional equations and inequalities, Math. Appl., vol. **518**, Kluwer Acad. Publ., Dordrecht, 2000, 65-75.

[3] M. Czerni, *General continuous solution of a nonlinear functional inequality*, Ann. Acad. Pedagog. Crac. Stud. Math. **1** (2001), 19-29.

[4] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne, Tom **46**, Państwowe Wydawnictwo Naukowe, Warsaw, 1968.

Institute of Mathematics
Pedagogical University,
Podchorążych 2
30-084, Kraków
Poland
E-mail: mczerni@ap.krakow.pl

Received: 2 July 2007; final version: 3 April 2008;
available online: 28 May 2008.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Mikhail Borsuk

The transmission problem for elliptic second order equations in a conical domain

Abstract. The present article is a survey of our last results. We establish best possible estimates of the weak solutions to the transmission problem near conical boundary point. We study this problem for the Laplace operator with N different media, for linear and quasi-linear (with semi-linear principal part) elliptic second order equations in divergence form. Boundary conditions in these problems are different: the Dirichlet, the Neumann, the Robin, as well as mixed boundary conditions.

The transmission problems often appear in different fields of physics and technics. For instance, one of the important problems of the electrodynamics of solid media is the electromagnetic processes research in ferromagnetic media with different dielectric constants. Such problems also appear in solid mechanics if a body consists of composite materials. Let us quote also vibrating folded membranes, composite plates, folded plates, junctions in elastic multi-structures etc.

The present article is a survey of our last results. We consider the best possible estimates of the weak solutions to the transmission problem near conical boundary point. Analogous results were established in [3] for the Dirichlet and Robin problems in a conical domain without interfaces.

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a conical surface with vertex at $\mathcal{O}$ and the opening ω_0. We assume that $G = \bigcup_{i=1}^{N} G_i$ is divided into $N \geq 2$ subdomains G_i, $i = 1, \ldots, N$ by $(N-1)$ hyperplanes Σ_k, $k = 1, \ldots, N-1$ (by hyperplane Σ_0 in the case $N = 2$), where $\mathcal{O}$ belongs to every $\overline{\Sigma_k}$ and $G_i \cap G_j = \emptyset$, $i \neq j$. We shall study the following elliptic transmission problems.

AMS (2000) Subject Classification: 35J25, 35J60, 35J85, 35B65.

This work was supported at the final stage by the Polish Ministry of Science and Higher Education through the grant Nr N201 381834..

Problem (*LN*). For the Laplace operator with N different media and mixed boundary condition

$$\begin{cases} \mathcal{L}_i[u] \equiv a_i \triangle u_i - p_i u_i(x) = f_i(x), & x \in G_i\,,\ i = 1, \dots, N; \\ [u]_{\Sigma_k} = 0, & k = 1, \dots, N-1; \\ \mathcal{S}_k[u] \equiv \left[a \dfrac{\partial u}{\partial \vec{n}_k} \right]_{\Sigma_k} + \dfrac{1}{|x|} \beta_k(\omega) u(x) = h_k(x), & x \in \Sigma_k\,, \\ & k = 1, \dots, N-1; \\ \mathcal{B}[u] \equiv \alpha(x) a \dfrac{\partial u}{\partial \vec{n}} + \dfrac{1}{|x|} \gamma(\omega) u(x) = g(x), & x \in \partial G \setminus \mathcal{O}, \end{cases}$$

where $\omega = \frac{x}{|x|}$, $a_i > 0$, $p_i \geq 0$, $(i = 1, \dots, N)$ are constants;

$$\alpha(x) = \begin{cases} 0, & \text{if } x \in \mathcal{D}; \\ 1, & \text{if } x \notin \mathcal{D}, \end{cases}$$

and $\mathcal{D} \subseteq \partial G$ is the part of the boundary ∂G where we consider the Dirichlet boundary condition; here $\vec{n}_k$ ($\vec{n}$) denotes the unite outward with respect to G_k (G) normal to Σ_k ($\partial G \setminus \mathcal{O}$).

Problem (*L*). For linear equations

$$\begin{cases} \mathcal{L}[u] \equiv \dfrac{\partial}{\partial x_i}(a^{ij}(x) u_{x_j}) + a^i(x) u_{x_i} + a(x) u = f(x), & x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0; & \\ \mathcal{S}[u] \equiv \left[\dfrac{\partial u}{\partial \nu} \right]_{\Sigma_0} + \dfrac{\beta(\omega)}{|x|} u(x) = h(x), & x \in \Sigma_0; \\ \mathcal{B}[u] \equiv \dfrac{\partial u}{\partial \nu} + \dfrac{\gamma(\omega)}{|x|} u = g(x), & x \in \partial G \setminus \mathcal{O}. \end{cases}$$

Problem (*WL*). For weak nonlinear equations

$$\begin{cases} -\dfrac{d}{dx_i}(|u|^q a^{ij}(x) u_{x_j}) + b(x, u, \nabla u) = 0, & q \geq 0,\ x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0; & \\ \mathcal{S}[u] \equiv \left[\dfrac{\partial u}{\partial \nu} \right]_{\Sigma_0} + \dfrac{\beta(\omega)}{|x|} u|u|^q = h(x, u), & x \in \Sigma_0; \\ \mathcal{B}[u] \equiv \dfrac{\partial u}{\partial \nu} + \dfrac{\gamma(\omega)}{|x|} u|u|^q = g(x, u), & x \in \partial G \setminus \mathcal{O} \end{cases}$$

(the summation over repeated indices from 1 to n is understood; $\frac{\partial u}{\partial \nu}$ is the co-normal derivative of $u(x)$), i.e., $\frac{\partial u}{\partial \nu} = |u|^q a^{ij}(x) u_{x_j} \cos(\vec{n}, x_i)$.

The principal new feature of our work is the consideration of estimates of weak solutions for *linear* elliptic second-order equations with *minimal smooth coefficients* in *n-dimensional conical* domains. Our examples demonstrate this fact.

1. Problem (LN)

Let ϕ_i be openings at the vertex $\mathcal{O}$ in domains G_i. Let us define the value $\theta_k = \phi_1 + \phi_2 + \ldots + \phi_k$, thus $\omega_0 = \theta_N$. We introduce the following notations:

— Ω_i – a domain on the unit sphere S^{n-1} with boundary $\partial\Omega_i$ obtained by the intersection of the domain G_i with the sphere S^{n-1}, $(i = 1, \ldots, N)$; thus $\Omega = \bigcup_{i=1}^{N} \Omega_i$;

— $\Sigma = \sum_{k=1}^{N-1} \Sigma_k$, $\quad \Sigma_k = G \cap \{\omega_1 = \frac{\omega_0}{2} - \theta_k\}$, $\qquad k = 1, \ldots, N-1$;
$\sigma = \sum_{k=1}^{N-1} \sigma_k$, $\quad \sigma_k = \Sigma_k \cap \Omega$;

— $(G_i)_a^b = \{(r, \omega) \mid \ 0 \le a < r < b;\ \omega \in \Omega\} \cap G_i \qquad i = 1, \ldots, N$;

— $(\Sigma_k)_a^b = G_a^b \cap \Sigma_k$, $\qquad k = 1, \ldots, N-1$;

— $u(x) = u_i(x)$, $\quad f(x) = f_i(x)$, $\quad x \in G_i$; $\quad a\big|_{\overline{G_i}} = a_i$, etc.;

— $[u]_{\Sigma_k}$ denotes the saltus of the function $u(x)$ on crossing Σ_k, i.e.,

$$[u]_{\Sigma_k} = u_k(\overline{x})\big|_{\Sigma_k} - u_{k+1}(\overline{x})\big|_{\Sigma_k},$$
$$u_k(\overline{x})\big|_{\Sigma_k} = \lim_{G_k \ni x \to \overline{x} \in \Sigma_k} u(x), \qquad u_{k+1}(\overline{x})\big|_{\Sigma_k} = \lim_{G_{k+1} \ni x \to \overline{x} \in \Sigma_k} u(x);$$

— $\left[a\frac{\partial u}{\partial \vec{n}_k}\right]_{\Sigma_k}$ denotes the saltus of the co-normal derivative of the function $u(x)$ on crossing Σ_k, i.e.,

$$\left[a\frac{\partial u}{\partial \vec{n}_k}\right]_{\Sigma_k} = a_k \frac{\partial u_k}{\partial \vec{n}_k}\bigg|_{\Sigma_k} - a_{k+1}\frac{\partial u_{k+1}}{\partial \vec{n}_k}\bigg|_{\Sigma_k}.$$

Without loss of generality we assume that there exists $d > 0$ such that G_0^d is *a convex rotational cone* with the vertex at $\mathcal{O}$ and the aperture ω_0, thus

$$\Gamma_0^d = \left\{(r, \omega) \;\Big|\; x_1^2 = \cot^2 \frac{\omega_0}{2} \sum_{i=2}^{n} x_i^2 \,;\ r \in (0, d),\ \omega_1 = \frac{\omega_0}{2},\ \omega_0 \in (0, \pi)\right\};$$

$\Gamma_a^b = \{(r, w) \mid \ 0 \le a < r < b;\ w \in \partial\Omega\} \cap \partial G$ – the lateral surface of layer G_a^b.

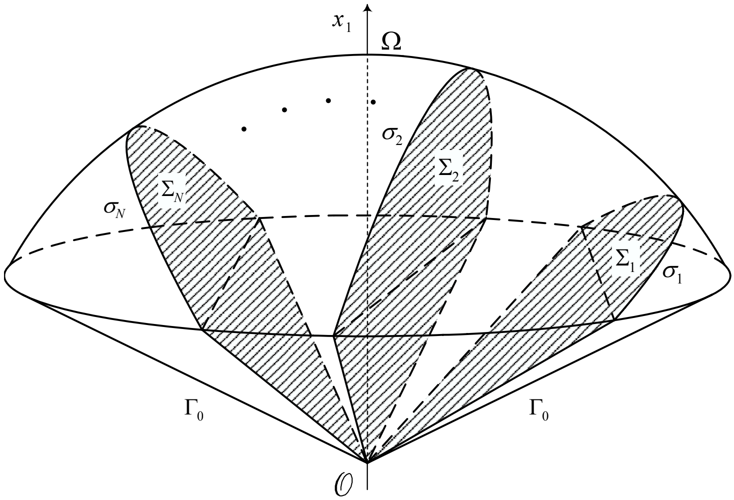

Fig. 1

We use the standard function spaces:

— $C^k(\overline{G_i})$ with the norm $|u_i|_{k,G_i}$,

— the Lebesgue space $L_p(G_i)$, $p \geq 1$ with the norm $\|u_i\|_{p,G_i}$,

— the Sobolev space $W^{k,p}(G_i)$ with the norm $\|u_i\|_{k,p;G_i}$,

— direct sum $\mathbf{C}^k(\overline{G}) = C^k(\overline{G_1}) \dotplus \ldots \dotplus C^k(\overline{G_N})$ with the norm

$$|u|_{k,G} = \sum_{i=1}^{N} |u_i|_{k,G_i} ;$$

— $\mathbf{L}_p(G) = L_p(G_1) \dotplus \ldots \dotplus L_p(G_N)$ with the norm

$$\|u\|_{\mathbf{L}_p(G)} = \sum_{i=1}^{N} \Big(\int_{G_i} |u_i|^p dx \Big)^{\frac{1}{p}};$$

— $\mathbf{W}^{k,p}(G) = W^{k,p}(G_1) \dotplus \ldots \dotplus W^{k,p}(G_N)$ with the norm

$$\|u\|_{k,p;G} = \sum_{i=1}^{N} \Big(\int_{G_i} \sum_{|\beta|=0}^{k} |D^\beta u_i|^p dx \Big)^{\frac{1}{p}}.$$

We define the weighted Sobolev spaces: $\mathbf{V}^k_{p,\alpha}(G) = V^k_{p,\alpha}(G_1) \dotplus \ldots \dotplus V^k_{p,\alpha}(G_N)$ for integer $k \geq 0$ and real α, where $V^k_{p,\alpha}(G_i)$ denotes the space of all distributions

$u \in \mathcal{D}'(G_i)$ satisfying $r^{\frac{\alpha}{p}+|\beta|-k}|D^\beta u_i| \in L_p(G_i)$, $i = 1, \dots, N$. $\mathbf{V}^k_{p,\alpha}(G)$ is a Banach space with the norm

$$\|u\|_{\mathbf{V}^k_{p,\alpha}(G)} = \sum_{i=1}^{N} \Bigg(\int\limits_{G_i} \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)} |D^\beta u_i|^p \, dx \Bigg)^{\frac{1}{p}}.$$

$\mathbf{V}^{k-\frac{1}{p}}_{p,\alpha}(\partial G)$ is the space of functions φ, given on ∂G, with the norm

$$\|\varphi\|_{\mathbf{V}^{k-\frac{1}{p}}_{p,\alpha}(\partial G)} = \inf \|\Phi\|_{\mathbf{V}^k_{p,\alpha}(G)},$$

where the infimum is taken over all functions Φ such that $\Phi\big|_{\partial G} = \varphi$ in the sense of traces. We denote $\mathbf{W}^k(G) \equiv \mathbf{W}^{k,2}(G)$, $\overset{\circ}{\mathbf{W}}{}^k_\alpha(G) \equiv \mathbf{V}^k_{2,\alpha}(G)$.

Definition 1
The function $u(x)$ is called a *weak* solution of the problem (LN) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_0$ and satisfies the integral identity

$$\int\limits_G a u_{x_j} \eta_{x_j} \, dx + \int\limits_\Sigma \frac{1}{r} \beta(\omega) u(x) \eta(x) \, ds + \int\limits_{\partial G} \alpha(x) \frac{1}{r} \gamma(\omega) u(x) \eta(x) \, ds$$
$$= \int\limits_{\partial G} \alpha(x) g(x) \eta(x) \, ds + \int\limits_\Sigma h(x) \eta(x) \, ds - \int\limits_G \left(p u(x) + f(x)\right) \eta(x) \, dx$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_0(G)$. The integrals above are sums:

$$\int\limits_G f(x) \, dx = \sum_{i=1}^{N} \int\limits_{G_i} f_i(x) \, dx, \qquad \int\limits_\Sigma h(x) \, ds = \sum_{k=1}^{N-1} \int\limits_{\Sigma_k} h_k(x) \, ds, \quad \text{etc.}$$

Remark 1
In the Dirichlet boundary condition case ($\alpha(x) \equiv 0$) we assume, without loss of generality, that

$$g\big|_{\partial G \cap \mathcal{D}} = 0 \implies u\big|_{\partial G \cap \mathcal{D}} = 0.$$

We assume that $M_0 = \max_{x \in \overline{G}} |u(x)|$ is known. Let us define numbers

$$\begin{cases} a_* = \min\{a_1, \dots, a_N\} > 0; & \\ a^* = \max\{a_1, \dots, a_N\} > 0; & \\ p^* = \max\{p_1, \dots, p_N\} \geq 0; & \\ [a]_{\Sigma_k} = a_k - a_{k+1}\,, & k = 1, \dots, N-1; \\ a_0 = \max\limits_{1 \leq k \leq N-1} \left|[a]_{\Sigma_k}\right|; & \\ \widetilde{a} = \max(a^*, a_0). & \end{cases}$$

We assume that:

(a) $f(x) \in \mathbf{L}_{\frac{q}{2}}(G) \cap \mathbf{L}_2(G)$; $q > n$;

(b) $\gamma(\phi) \geq \gamma_0 > \widetilde{a} \tan \frac{\omega_0}{2}$ *on* ∂G;
$\beta_k(\phi) \geq \beta_0 > \widetilde{a} \tan \frac{\omega_0}{2}$ *on* Σ_k, $k = 1, \ldots, N-1$;

(c) *there exist numbers* $f_0 \geq 0$, $g_0 \geq 0$, $h_0 \geq 0$, $s > 1$, $\beta \geq s - 2$ *such that*

$$|f(x)| \leq f_0 |x|^{\beta}, \qquad |g(x)| \leq g_0 |x|^{s-1},$$
$$|h_k(x)| \leq h_0 |x|^{s-1}, \quad k = 1, \ldots, N-1.$$

We consider the following **eigenvalue problem** $(\boldsymbol{EVP})$.

Let $\Omega \subset S^{n-1}$ with smooth boundary $\partial\Omega$ be the intersection of the cone $\mathcal{C}$ with the unit sphere S^{n-1}. Let $\vec{\nu}$ be the exterior normal to $\partial\mathcal{C}$ at points of $\partial\Omega$ and $\vec{\tau}_k$ be the exterior with respect to Ω_k normal to Σ_k (lying in the plane tangent to Ω_k), $k = 1, \ldots, N-1$. Let $\gamma(\phi)$, $\phi \in \partial\Omega$ be a positive bounded piecewise smooth function, $\beta_k(\phi)$ be a positive continuous function on σ_k, $k = 1, \ldots, N-1$. We consider the eigenvalue problem for the Laplace-Beltrami operator $\triangle_\phi$ on the unit sphere

$$\begin{cases} a_i \left(\triangle_\phi \psi_i + \vartheta \psi_i \right) = 0, & \phi \in \Omega_i, \ a_i \text{ are positive constants; } i = 1, \ldots, N; \\ [\psi]_{\sigma_k} = 0, & k = 1, \ldots, N-1; \\ \left[a \dfrac{\partial \psi}{\partial \vec{\tau}_k} \right]_{\sigma_k} + \beta_k(\phi)\psi\big|_{\sigma_k} = 0, & k = 1, \ldots, N-1; \\ \alpha(\phi) a \dfrac{\partial \psi}{\partial \vec{\nu}} + \gamma(\phi)\psi\big|_{\partial\Omega} = 0, & \end{cases} \qquad (EVP)$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has a non-zero weak solutions (eigenfunctions).

Our main result is the following theorem. Let ϑ be the smallest positive solution of (EVP) and let

$$\lambda = \frac{2 - n + \sqrt{(n-2)^2 + 4\vartheta}}{2}. \qquad (1.1)$$

THEOREM 1
Let u be a weak solution of the problem (LN) and assumptions (a)-(c) *be satisfied. Assume that the domain G and parameters in* (a)-(c) *are such that λ defined above satisfies $\lambda > 1$. Then there are $d \in (0,1)$ and constants $C_0 > 0$, $c > 0$ depending only on n, a_*, a^*, p^*, λ, q, ω_0, f_0, h_0, g_0, β_0, γ_0, s, M_0,* meas G, diam G *such that for all $x \in G_0^d$*

$$|u(x)| \le C_0 \begin{cases} |x|^{\lambda}, & \textit{if } s > \lambda; \\ |x|^{\lambda} \ln^{c}\left(\dfrac{1}{|x|}\right), & \textit{if } s = \lambda; \\ |x|^{s}, & \textit{if } s < \lambda. \end{cases}$$

Suppose, in addition, that

$$\gamma(\omega) \in \mathbf{C}^1(\partial G), \qquad f(x) \in \mathbf{V}^0_{q,2q-n}(G),$$
$$h(x) \in V^{1-\frac{1}{q}}_{q,2q-n}(\Sigma), \qquad g(x) \in \mathbf{V}^{1-\frac{1}{q}}_{q,2q-n}(\partial G);$$

$q > n$ and there is a number

$$\tau_s =: \sup_{\varrho>0} \varrho^{-s}\Big(\|h\|_{V^{1-\frac{1}{q}}_{q,2q-n}(\Sigma^{\varrho}_{\frac{\varrho}{2}})} + \|g\|_{\mathbf{V}^{1-\frac{1}{q}}_{q,2q-n}(\Gamma^{\varrho}_{\frac{\varrho}{2}})}\Big).$$

Then for all $x \in G^d_0$

$$|\nabla u(x)| \le C_1 \begin{cases} |x|^{\lambda-1}, & \textit{if } s > \lambda; \\ |x|^{\lambda-1} \ln^{c}\left(\dfrac{1}{|x|}\right), & \textit{if } s = \lambda; \\ |x|^{s-1}, & \textit{if } s < \lambda. \end{cases}$$

Furthermore, the following is true

— *$u \in \mathbf{V}^2_{q,2q-n}(G)$, $q > n$ and*

$$\|u\|_{\mathbf{V}^2_{q,2q-n}(G^{\varrho}_0)} \le C_2 \begin{cases} \varrho^{\lambda}, & \textit{if } s > \lambda; \\ \varrho^{\lambda} \ln^{c}\left(\dfrac{1}{\varrho}\right), & \textit{if } s = \lambda; \\ \varrho^{s}, & \textit{if } s < \lambda; \end{cases}$$

— *if $f(x) \in \mathring{\mathbf{W}}^0_{\alpha}(G)$, $\int_{\Sigma} r^{\alpha-1} h^2(x)\, ds + \int_{\partial G} r^{\alpha-1} g^2(x)\, ds < \infty$, where $4 - n - 2\lambda < \alpha \le 2$, then $u(x) \in \mathring{\mathbf{W}}^1_{\alpha-2}(G)$ and*

$$\int_G a(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)\, dx + \int_{\Sigma} r^{\alpha-3}\beta(\phi) u^2(x)\, ds$$
$$+ \int_{\partial G} \alpha(x) r^{\alpha-3}\gamma(\phi) u^2(x)\, ds$$
$$\le C\Big\{ \int_G (u^2 + (1 + r^{\alpha}) f^2(x))\, dx + \int_{\Sigma} r^{\alpha-1} h^2(x)\, ds$$
$$+ \int_{\partial G} \alpha(x) r^{\alpha-1} g^2(x)\, ds \Big\},$$

where the constant $C > 0$ depends only on q, n, a_*, a^*, α, λ and the domain G.

1.1. Eigenvalue transmission problem in a composite plane domain with an angular point

Let $G \subset \mathbb{R}^2$ be bounded domain with the boundary curve ∂G smooth everywhere except at the origin $\mathcal{O} \in \partial G$. Near the point $\mathcal{O}$ it is a fan that consists of N corners with vertices at $\mathcal{O}$. Thus $G = \bigcup_{i=1}^{N} G_i$; $\partial G = \bigcup_{j=0}^{N+1} \Gamma_j$; $\Sigma = \bigcup_{k=1}^{N-1} \Sigma_k$. Here Σ_k, $k = 1, \ldots, N-1$ are rays that divide G into angular domains G_i, $i = 1, \ldots, N$. Let ω_i be apertures at the vertex $\mathcal{O}$ in domains G_i, $i = 1, \ldots, N$. We define the value $\theta_k = \omega_1 + \omega_2 + \ldots + \omega_k$. Let $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ be the curvilinear portion of the boundary ∂G. In this case we have $\lambda = \sqrt{\vartheta}$.

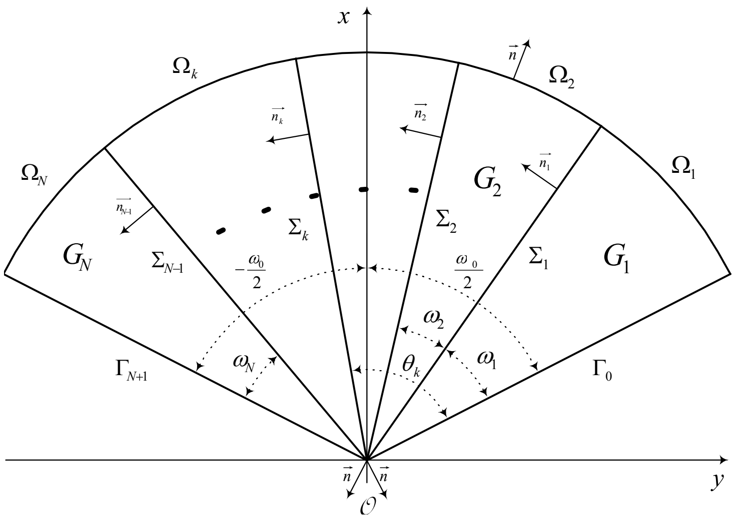

Fig. 2

We also assume that

$$\Gamma_0 = \{(r, \omega) \mid r > 0,\ \omega = 0\}, \qquad \Gamma_{N+1} = \{(r, \omega) \mid r > 0,\ \omega = \theta_N\},$$

$$\beta_k\big|_{\sigma_k} = \beta_k(\theta_k) = \beta_k = \text{const}, \quad \gamma(0) = \gamma_1 = \text{const}, \quad \gamma(\omega_0) = \gamma_N = \text{const}.$$

The eigenvalue problem in this case has the form

$$\begin{cases} \psi_i'' + \lambda^2 \psi_i(\omega) = 0, & \omega \in \Omega_i = \{\omega_{i-1} < \omega < \omega_i\}, \\ & i = 1, \dots, N; \\ \psi_i(\theta_i) = \psi_{i+1}(\theta_i), & i = 1, \dots, N-1; \\ a_i \psi_i'(\theta_i) - a_{i+1}\psi_{i+1}'(\theta_i) + \beta_i \psi_i(\theta_i) = 0, & i = 1, \dots, N-1; \\ \alpha_1 a_1 \psi_1'(0) + \gamma_1 \psi_1(0) = 0; \\ \alpha_N a_N \psi_N'(\omega_0) + \gamma_N \psi_N(\omega_0) = 0, \end{cases}$$

where $a_1, \alpha_N \in \{0, 1\}$.

By direct calculation, we get

$$\psi_i(\omega) = A_i \cos(\lambda\omega) + B_i \sin(\lambda\omega), \qquad i = 1, \dots, N$$

and constants $A_1, \dots, A_N$; $B_1, \dots, B_N$ are defined by the algebraic homogeneous system of equations:

$$\begin{cases} \lambda \alpha_1 a_1 B_1 + \gamma_1 = 0; \\ A_{i+1} = \left(\cos^2(\lambda\theta_i) + \dfrac{a_i}{a_{i+1}} \sin^2(\lambda\theta_i) - \dfrac{\beta_i}{\lambda a_{i+1}} \sin(\lambda\theta_i)\cos(\lambda\theta_i)\right) \cdot A_i \\ \qquad + \left(\sin(\lambda\theta_i)\cos(\lambda\theta_i)\left(1 - \dfrac{a_i}{a_{i+1}}\right) - \dfrac{\beta_i}{\lambda a_{i+1}} \sin^2(\lambda\theta_i)\right) \cdot B_i, \\ \qquad i = 1, \dots, N-1; \\ B_{i+1} = \left(\sin(\lambda\theta_i)\cos(\lambda\theta_i)\left(1 - \dfrac{a_i}{a_{i+1}}\right) + \dfrac{\beta_i}{\lambda a_{i+1}} \cos^2(\lambda\theta_i)\right) \cdot A_i \\ \qquad + \left(\sin^2(\lambda\theta_i) + \dfrac{a_i}{a_{i+1}} \cos^2(\lambda\theta_i) + \dfrac{\beta_i}{\lambda a_{i+1}} \sin(\lambda\theta_i)\cos(\lambda\theta_i)\right) \cdot B_i, \\ \qquad i = 1, \dots, N-1; \\ (\gamma_N \cos(\lambda\omega_0) - \lambda\alpha_N a_N \sin(\lambda\omega_0)) \cdot A_N + (\gamma_N \sin(\lambda\omega_0) \\ \qquad + \lambda \alpha_N a_N \cos(\lambda\omega_0)) \cdot B_N = 0. \end{cases}$$

The least positive eigenvalue λ is determined by the vanishing of the determinant of this system.

1.2. Four media transmission problem

Our goal is to derive the eigenvalues equation corresponding to our transmission problem for $N = 4$. Let S^1 be the unit circle in $\mathbb{R}^2$. We denote: $\Omega_i = G_i \cap S^1$ for $i = 1, 2, 3, 4$. The eigenvalue problem is the following:

$$\begin{cases} \psi_i'' + \lambda^2\psi_i(\omega) = 0, & \omega \in \Omega_i\,;\ \text{for } i = 1,2,3,4; \\ \psi_2(\omega_1) = \psi_1(\omega_1); \\ \psi_3(\theta_2) = \psi_2(\theta_2); \\ \psi_4(\theta_3) = \psi_3(\theta_3); \\ a_1\psi_1'(\omega_1) - a_2\psi_2'(\omega_1) + \beta_1\psi_1(\omega_1) = 0; \\ a_2\psi_2'(\theta_2) - a_3\psi_3'(\theta_2) + \beta_2\psi_2(\theta_2) = 0; \\ a_3\psi_3'(\theta_3) - a_4\psi_4'(\theta_3) + \beta_3\psi_3(\theta_3) = 0; \\ \alpha_1 a_1\psi_1'(0) + \gamma_1\psi_1(0) = 0; \\ \alpha_4 a_4\psi_4'(\theta_4) + \gamma_4\psi_4(\theta_4) = 0, \end{cases} \tag{1.2}$$

where $\alpha_1 = \alpha\big|_{\Gamma_0} = \alpha\big|_{\omega=0}$, $\alpha_4 = \alpha\big|_{\Gamma_5} = \alpha\big|_{\omega=\theta_4}$, $\gamma_1 = \gamma(0)$, $\gamma_4 = \gamma(\theta_4)$; $\alpha_{1,4} \in \{0,1\}$.

A general solution of (1.2) is

$$\psi_i(\omega) = A_i\cos(\lambda\omega) + B_i\sin(\lambda\omega) \qquad \text{for } i = 1,2,3,4,$$

with arbitrary constants A_i, B_i ($i = 1,2,3,4$). Boundary condition of (1.2) force A_i, B_i to satisfy the following system of linear equations:

$$\begin{cases} A_2\cos\lambda\omega_1 + B_2\sin\omega_1 - A_1\cos\lambda\omega_1 - B_1\sin\lambda\omega_1 = 0, \\ A_3\cos\lambda\theta_2 + B_3\sin\theta_2 - A_2\cos\lambda\theta_2 - B_2\sin\lambda\theta_2 = 0, \\ A_4\cos\lambda\theta_3 + B_4\sin\theta_3 - A_3\cos\lambda\theta_3 - B_3\sin\lambda\theta_3 = 0, \\ \lambda a_2A_2\sin\lambda\omega_1 - \lambda a_2B_2\cos\lambda\omega_1 - \lambda a_1A_1\sin\lambda\omega_1 + \lambda a_1B_1\cos\lambda\omega_1 \\ \qquad +\beta_1A_1\cos\lambda\omega_1 + \beta_1B_1\sin\lambda\omega_1 = 0, \\ \lambda a_3A_3\sin\lambda\theta_2 - \lambda a_3B_3\cos\lambda\theta_2 - \lambda a_2A_2\sin\lambda\theta_2 + \lambda a_2B_2\cos\lambda\theta_2 \\ \qquad +\beta_2A_2\cos\lambda\theta_2 + \beta_2B_2\sin\lambda\theta_2 = 0, \\ \lambda a_4A_4\sin\lambda\theta_3 - \lambda a_4B_4\cos\lambda\theta_3 - \lambda a_3A_3\sin\lambda\theta_3 + \lambda a_3B_3\cos\lambda\theta_3 \\ \qquad +\beta_3A_3\cos\lambda\theta_3 + \beta_3B_3\sin\lambda\theta_3 = 0, \\ \alpha_1a_1\lambda B_1 + \gamma_1A_1 = 0, \\ \alpha_4a_4\lambda A_4\sin\lambda\theta_4 - \alpha_4a_4\lambda B_4\cos\lambda\theta_4 - \gamma_4A_4\cos\lambda\theta_4 - \gamma_4B_4\sin\lambda\theta_4 = 0. \end{cases}$$

This system has a non-trivial solution if its determinant vanishes. This gives the eigenvalues equation, which is too complex to state here in full generality. We provide an explicite form only in special cases of boundary conditions.

1. Dirichlet problem: $\alpha_1 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0$; $\gamma_1 = \gamma_4 = 1$.

$$\begin{aligned} \lambda^3 a_2a_3^2 &\sin\lambda\omega_1\cos\lambda\omega_2\sin\lambda\omega_3\sin\lambda\omega_4 \\ &+ a_2^2a_3\sin\lambda\omega_1\sin\lambda\omega_2\cos\lambda\omega_3\sin\lambda\omega_4 \\ &+ a_2^2a_4\sin\lambda\omega_1\sin\lambda\omega_2\sin\lambda\omega_3\cos\lambda\omega_4 \\ &+ a_1a_3^2\cos\lambda\omega_1\sin\lambda\omega_2\sin\lambda\omega_3\sin\lambda\omega_4 \end{aligned}$$

$$
\begin{aligned}
&- a_2 a_3 a_4 \sin\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1 a_3 a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1 a_2 a_4 \cos\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1 a_2 a_3 \cos\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
= 0.&
\end{aligned}
$$

In the isotropic case ($a_1 = a_2 = a_3 = a_4$) we recover the well known result:

$$\sin(\lambda\theta_4) = 0 \implies \lambda_n = \frac{\pi n}{\theta_4}, \qquad n = 1, 2 \ldots .$$

Corollary: $\lambda = \frac{\pi}{\omega_0} > 1$, if $0 < \omega_0 < \pi$.

2. Neumann problem: $\alpha_1 = \alpha_4 = 1$; $\beta_1 = \beta_2 = \beta_3 = 0$; $\gamma_1 = \gamma_4 = 0$.

$$
\begin{aligned}
-a_1^2 a_2 a_4^2 &\sin\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2 a_3 a_4^2 \sin\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2 a_3^2 a_4 \sin\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1 a_2^2 a_4^2 \cos\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&+ a_1^2 a_2 a_3 a_4 \sin\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&+ a_1 a_2^2 a_3 a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&+ a_1 a_2 a_3^2 a_4 \cos\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&+ a_1 a_2 a_3 a_4^2 \cos\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
= 0.&
\end{aligned}
$$

In the isotropic case ($a_1 = a_2 = a_3 = a_4$) we recover again:

$$\sin(\lambda\theta_4) = 0 \implies \lambda_n = \frac{\pi n}{\theta_4}, \qquad n = 0, 1, 2 \ldots .$$

Corollary: $\lambda = \frac{\pi}{\omega_0} > 1$, if $0 < \omega_0 < \pi$.

3. Mixed problem: $\alpha_1 = \gamma_4 = 1$, $\alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0$; $\gamma_1 = 0$.

$$
\begin{aligned}
a_1^2 a_3^2 &\sin\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2 a_3 a_4 \sin\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1^2 a_2 a_3 \sin\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2 a_2 a_4 \sin\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1 a_2 a_3^2 \cos\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1 a_2^2 a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1 a_2^2 a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
&+ a_1 a_2 a_3 a_4 \cos\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
= 0.&
\end{aligned}
$$

In the isotropic case ($a_1 = a_2 = a_3 = a_4$) we hence recover:

$$\cos(\lambda\theta_4) = 0 \implies \lambda_n = \frac{\pi(2n-1)}{2\theta_4}, \qquad n = 1, 2 \dots .$$

COROLLARY: $\lambda = \frac{\pi}{2\omega_0} > 1$, if $0 < \omega_0 < \frac{\pi}{2}$.

4. ROBIN PROBLEM: $\alpha_1 = \alpha_4 = 1$.

   In the isotropic case ($a_1 = a_2 = a_3 = a_4 = 1$; $\beta_1 = \beta_2 = \beta_3 = 0$) we obtain:

$$\tan(\lambda\omega_0) = \frac{\lambda(\gamma_4 - \gamma_1)}{\lambda^2 + \gamma_1\gamma_4}.$$

1.3. Three media transmission problem

Our goal is the derivation of the eigenvalues equation that corresponds to our transmission problem for the case $N = 3$. Let S^1 be the unit circle in $\mathbb{R}^2$ centered at $\mathcal{O}$. We denote: $\Omega_i = G_i \cap S^1$; $i = 1, 2, 3$. The eigenvalue problem is the following one:

$$\begin{cases} \psi_i'' + \lambda^2\psi_i(\omega) = 0, & \omega \in \Omega_i\,;\ (i = 1,2,3); \\ \psi_1(\omega_1) = \psi_2(\omega_1);\ \psi_3(\theta_2) = \psi_2(\theta_2); \\ a_2\psi_2'(\omega_1) - a_1\psi_1'(\omega_1) + \beta_1\psi_1(\omega_1) = 0; \\ a_3\psi_3'(\theta_2) - a_2\psi_2'(\theta_2) + \beta_2\psi_2(\theta_2) = 0; \\ \alpha_1 a_1\psi_1'(0) + \gamma_1\psi_1(0) = 0; \\ \alpha_3 a_3\psi_3'(\theta_3) + \gamma_3\psi_3(\theta_3) = 0. \end{cases} \tag{1.3}$$

We find a general solution of (1.3):

$$\psi_i(\omega) = A_i\cos(\lambda\omega) + B_i\sin(\lambda\omega) \qquad \text{for } i = 1, 2, 3,$$

where A_i, B_i ($i = 1, 2, 3$) are arbitrary constants. From the boundary condition of (1.3) we obtain the homogenous algebraic system of six linear equations determining A_i , B_i ($i = 1, 2, 3$). The determinant of the system must be equal to zero for a nontrivial solution of this system to exist. The latter gives the required eigenvalues λ-equation:

$$
\begin{aligned}
&[\lambda^2\alpha_3 a_3^2(\beta_1\gamma_1+\lambda^2\alpha_1 a_1^2)-\gamma_3(\beta_1\beta_2\gamma_1+\lambda^2\alpha_1\beta_2 a_1^2-\lambda^2\gamma_1 a_2^2)]\\
&\qquad\times\sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\qquad+\lambda a_1\cdot[\lambda^2\alpha_3 a_3^2(\gamma_1-\beta_1\alpha_1)+\gamma_3(\beta_1\beta_2\alpha_1-\gamma_1\beta_2-\lambda^2\alpha_1 a_2^2)]\\
&\qquad\times\cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\qquad-\lambda a_3\cdot[\gamma_3(\beta_1\gamma_1+\lambda^2\alpha_1 a_1^2)+\alpha_3(\beta_1\beta_2\gamma_1+\lambda^2\alpha_1\beta_2 a_1^2-\lambda^2\gamma_1 a_2^2)]\\
&\qquad\times\sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad+\lambda^2 a_1 a_3\cdot[\gamma_3(\beta_1\alpha_1-\gamma_1)+\alpha_3(\beta_1\beta_2\alpha_1-\gamma_1\beta_2-\lambda^2\alpha_1 a_2^2)]\\
&\qquad\times\cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad-\lambda a_2\cdot[\gamma_3(\beta_2\gamma_1+\lambda^2\alpha_1 a_1^2+\beta_1\gamma_1)-\lambda^2\alpha_3\gamma_1 a_3^2]\\
&\qquad\times\sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\qquad+\lambda^2 a_1 a_2\cdot[\gamma_3(\beta_2\alpha_1+\alpha_1\beta_1-\gamma_1)-\lambda^2\alpha_3\alpha_1 a_3^2]\\
&\qquad\times\cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\qquad-\lambda^2 a_2 a_3\cdot[\gamma_1\gamma_3+\alpha_3(\beta_2\gamma_1+\lambda^2\alpha_1 a_1^2+\beta_1\gamma_1)]\\
&\qquad\times\sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad+\lambda^3 a_1 a_2 a_3\cdot[\alpha_1\gamma_3+\alpha_3(\beta_2\alpha_1+\alpha_1\beta_1-\gamma_1)]\\
&\qquad\times\cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\quad=0.
\end{aligned}
\tag{1.4}
$$

We consider special cases of boundary conditions.

1. Dirichlet problem: $\alpha_1=\alpha_3=\beta_1=\beta_2=0$; $\gamma_1=\gamma_3=1$.

$$
\begin{aligned}
a_1a_3\cdot&\cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&+a_1a_2\cdot\cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&+a_2a_3\cdot\sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&-a_2^2\cdot\sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
=\;&0.
\end{aligned}
$$

In the isotropic case ($a_1=a_2=a_3$) we obtain the well known result:

$$\sin(\lambda\theta_3)=0 \implies \lambda_n=\frac{\pi n}{\theta_3},\qquad n=1,2,\dots\ .$$

Corollary: $\lambda=\frac{\pi}{\theta_3}>1$, if $\theta_3=\omega_1+\omega_2+\omega_3<\pi$.

2. Neumann problem: $\beta_1=\beta_2=\gamma_1=\gamma_3=0$; $\alpha_1=\alpha_3=1$.

$$
\begin{aligned}
a_2^2\cdot&\cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&+a_2a_3\cdot\cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&+a_1a_2\cdot\sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&-a_1a_3\cdot\sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
=\;&0.
\end{aligned}
$$

In the isotropic case ($a_1=a_2=a_3$) we hence obtain:

$$\sin(\lambda\theta_3) = 0 \implies \lambda_n = \frac{\pi n}{\theta_3}, \qquad n = 0, 1, 2, \dots .$$

Corollary: $\lambda = \frac{\pi}{\theta_3} > 1$, if $\theta_3 = \omega_1 + \omega_2 + \omega_3 < \pi$.

3. Mixed problem: $\alpha_1 = \gamma_3 = 1$, $\alpha_3 = \beta_1 = \beta_2 = \gamma_1 = 0$.

$$\begin{aligned} a_2^2 \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) & \\ + a_1 a_3 \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) & \\ + a_1 a_2 \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - a_2 a_3 \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) & \\ = 0. & \end{aligned}$$

In the isotropic case ($a_1 = a_2 = a_3$):

$$\cos(\lambda\theta_3) = 0 \implies \lambda_n = \frac{\pi(2n-1)}{2\theta_3}, \qquad n = 1, 2, \dots .$$

Corollary: $\lambda = \frac{\pi}{2\theta_3} > 1$, if $\theta_3 = \omega_1 + \omega_2 + \omega_3 < \frac{\pi}{2}$.

4. Robin problem: $\alpha_1 = 1$, $\alpha_3 = 1$; $\beta_1 = \beta_2 = 0$.

$$\begin{aligned} (\lambda^2 a_1^2 a_3^2 + \gamma_1\gamma_3 a_2^2) \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - \lambda \cdot (\gamma_3 a_1 a_2^2 - \gamma_1 a_1 a_3^2) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - \lambda a_3 \cdot (\gamma_3 a_1^2 - \gamma_1 a_2^2) \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) & \\ - a_1 a_3 (\gamma_1\gamma_3 + \lambda^2 a_2^2) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) & \\ - \lambda a_2 \cdot (\gamma_3 a_1^2 - \gamma_1 a_3^2) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - a_1 a_2 \cdot (\gamma_1\gamma_3 + \lambda^2 a_3^2) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - a_2 a_3 \cdot (\gamma_1\gamma_3 + \lambda^2 a_1^2) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) & \\ + \lambda a_1 a_2 a_3 \cdot (\gamma_3 - \gamma_1) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) & \\ = 0. & \end{aligned}$$

In the isotropic case ($a_1 = a_2 = a_3 = 1$) we recover (see [3], §10.1.7, Example 1):

$$\tan(\lambda\theta_3) = \frac{\lambda(\gamma_3 - \gamma_1)}{\lambda^2 + \gamma_1\gamma_3}.$$

1.4. Two media transmission problem

Here we consider in detail 2-dimensional transmission problem with two different media ($\omega_1 = \omega_2 = \frac{\omega_0}{2}$) for the Laplace operator in an angular symmetric domain and investigate the corresponding eigenvalue problem. Suppose $n = 2$, the domain G lies inside the angle

$$G_0 = \left\{ (r, \omega) \,\middle|\, r > 0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2} \right\}, \qquad \omega_0 \in \,]0, 2\pi[\,;$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_\pm = \left\{(r,\omega) \,\middle|\, r > 0;\ \omega = \pm\frac{\omega_0}{2}\right\}, \qquad \Sigma_0 = \{(r,\omega) \mid r > 0;\ \omega = 0\}$$

and we put

$$\beta(\omega)\big|_{\Sigma_0} = \beta(0) = \beta = \mathrm{const} \geq 0, \qquad \gamma(\omega)\big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = \mathrm{const} > 0.$$

We consider the following problem:

$$\begin{cases} a_\pm \triangle u_\pm = f_\pm(x), & x \in G_\pm; \\ [u]_{\Sigma_0} = 0; & \\ \left[a\dfrac{\partial u}{\partial \vec{n}}\right]_{\Sigma_0} + \dfrac{1}{|x|}\beta u(x) = h(x), & x \in \Sigma_0; \\ \alpha_\pm a_\pm \dfrac{\partial u_\pm}{\partial \vec{n}} + \dfrac{1}{r}\gamma_\pm u_\pm(x) = g_\pm(x), & x \in \Gamma_\pm \setminus \mathcal{O}. \end{cases} \tag{1.5}$$

It is well known that the homogeneous problem $(f(x) = h(x) = g(x) = 0)$ has solution of the form $u(r,\omega) = r^\lambda \psi(\omega)$, where λ is an eigenvalue and $\psi(\omega)$ is the corresponding eigenfunction of the problem

$$\begin{cases} \psi_+'' + \lambda^2 \psi_+(\omega) = 0, & \text{for } \omega \in (0, \frac{\omega_0}{2}); \\ \psi_-'' + \lambda^2 \psi_-(\omega) = 0, & \text{for } \omega \in (-\frac{\omega_0}{2}, 0); \\ \psi_+(0) = \psi_-(0); & \\ a_+\psi_+'(0) - a_-\psi_-'(0) = \beta\psi(0); & \\ \pm\alpha_\pm a_\pm \psi'\left(\pm\dfrac{\omega_0}{2}\right) + \gamma_\pm \psi\left(\pm\dfrac{\omega_0}{2}\right) = 0. & \end{cases} \tag{1.6}$$

The case $\lambda = 0$

In this case the solution of our equations has the form

$$\psi_\pm(\omega) = A_\pm \cdot \omega + B_\pm\,.$$

From the boundary conditions we obtain $B_+ = B_- = B$ and to find A_+, A_-, B, we have the system

$$\begin{cases} a_+A_+ - a_-A_- - \beta B = 0, \\ \left(\alpha_+ a_+ + \dfrac{\omega_0}{2}\gamma_+\right) A_+ + \gamma_+ B = 0, \\ -\left(\alpha_- a_- + \dfrac{\omega_0}{2}\gamma_-\right) A_- + \gamma_- B = 0. \end{cases}$$

Since $A_+^2 + A_-^2 + B^2 \neq 0$, the determinant must be equal to zero; this means

$$\beta\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right)\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right) + a_+\gamma_+\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right) + a_-\gamma_-\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right) = 0. \tag{1.7}$$

Thus if the equality (1.7) is satisfied, then $\lambda = 0$ and the corresponding eigenfunctions are

$$\psi(\omega) = \begin{cases} a_-\gamma_-\left\{\left(\omega - \frac{\omega_0}{2}\right)\gamma_+ - \alpha_+a_+\right\}, & \omega \in (0, \frac{\omega_0}{2}); \\ a_+\gamma_+\left\{\left(\omega + \frac{\omega_0}{2}\right)\gamma_- - \alpha_-a_-\right\}, & \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \quad \text{if } \beta = 0;$$

$$\psi(\omega) = \begin{cases} -\gamma_+\left(\alpha_-a_- + \frac{\omega_0\gamma_-}{2}\right)(\omega + \frac{a_+}{\beta}) - \frac{a_-\gamma_-}{\beta}\left(\alpha_+a_+ + \frac{\omega_0\gamma_+}{2}\right), \\ \qquad \omega \in (0, \frac{\omega_0}{2}); \\ \gamma_-\left(\alpha_+a_+ + \frac{\omega_0\gamma_+}{2}\right)(\omega - \frac{a_-}{\beta}) - \frac{a_+\gamma_+}{\beta}\left(\alpha_-a_- + \frac{\omega_0\gamma_-}{2}\right), \\ \qquad \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \quad \text{if } \beta \neq 0.$$

The case $\lambda \neq 0$

In this case the solution of our equations has the form

$$\psi_\pm(\omega) = A_\pm \cos(\lambda\omega) + B_\pm \sin(\lambda\omega).$$

From the boundary conditions we obtain $A_+ = A_- = A$ and to find A, B_+, B_- we have the system

$$\begin{cases} \beta A - \lambda a_+B_+ + \lambda a_-B_- = 0, \\ \left(\gamma_+ \cos\frac{\lambda\omega_0}{2} - \lambda\alpha_+a_+ \sin\frac{\lambda\omega_0}{2}\right)A + \left(\gamma_+ \sin\frac{\lambda\omega_0}{2} + \lambda\alpha_+a_+ \cos\frac{\lambda\omega_0}{2}\right)B_+ = 0, \\ \left(\gamma_- \cos\frac{\lambda\omega_0}{2} - \lambda\alpha_-a_- \sin\frac{\lambda\omega_0}{2}\right)A - \left(\gamma_- \sin\frac{\lambda\omega_0}{2} + \lambda\alpha_-a_- \cos\frac{\lambda\omega_0}{2}\right)B_- = 0. \end{cases}$$

Since $A^2 + B_+^2 + B_-^2 \neq 0$, the determinant must be zero; this means that λ is defined by the transcendental equation

$$\begin{aligned} &\beta(\lambda^2\alpha_+\alpha_-a_+a_- + \gamma_+\gamma_-) + \lambda^2(a_+ - a_-)(\alpha_-a_-\gamma_+ - \alpha_+a_+\gamma_-) \\ &\quad + \lambda[\beta(\alpha_-a_-\gamma_+ + \alpha_+a_+\gamma_-) \\ &\qquad + (a_+ + a_-)(\gamma_+\gamma_- - \lambda^2\alpha_+\alpha_-a_+a_-)]\sin(\lambda\omega_0) \\ &\quad + [\beta(\lambda^2\alpha_+\alpha_-a_+a_- - \gamma_+\gamma_-) \\ &\qquad + \lambda^2(a_+ + a_-)(\alpha_-a_-\gamma_+ + \alpha_+a_+\gamma_-)]\cos(\lambda\omega_0) \\ &= 0. \end{aligned} \tag{1.8}$$

Now we investigate special cases of the boundary conditions.

1. THE DIRICHLET PROBLEM: $\alpha\pm = 0$. Equation (1.8) takes the form

$$\beta(1-\cos(\lambda\omega_0)) + \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0.$$

Hence we get

$$\lambda = \begin{cases} \dfrac{\pi}{\omega_0}, & \text{if } \beta = 0; \\ \text{the least positive root of } \tan\dfrac{\lambda\omega_0}{2} = -\dfrac{a_+ + a_-}{\beta}\cdot\lambda, & \text{if } \beta \neq 0 \end{cases}$$

and the corresponding eigenfunction is

$$\psi(\omega) = \begin{cases} \sin\lambda\left(\dfrac{\omega_0}{2} - \omega\right), & \omega \in (0, \frac{\omega_0}{2}); \\ \sin\lambda\left(\dfrac{\omega_0}{2} + \omega\right), & \omega \in (-\frac{\omega_0}{2}, 0). \end{cases}$$

2. THE NEUMANN PROBLEM: $\gamma\pm = 0$. Equation (1.8) takes the form

$$\beta(1+\cos(\lambda\omega_0)) - \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0.$$

Hence we get $\lambda = \min\{\lambda^*, \frac{\pi}{\omega_0}\}$, where λ^* is the least positive root of the transcendental equation

$$\tan\frac{\lambda\omega_0}{2} = \frac{\beta}{a_+ + a_-}\cdot\frac{1}{\lambda}.$$

We find the corresponding eigenfunctions

$$\psi(\omega) = \begin{cases} a_- \sin\dfrac{\pi\omega}{\omega_0}, & \omega \in (0, \frac{\omega_0}{2}); \\ a_+ \sin\dfrac{\pi\omega}{\omega_0}, & \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \qquad \lambda = \frac{\pi}{\omega_0};$$

$$\psi(\omega) = \begin{cases} \cos\lambda^*\left(\omega - \dfrac{\omega_0}{2}\right), & \omega \in (0, \frac{\omega_0}{2}); \\ \cos\lambda^*\left(\omega + \dfrac{\omega_0}{2}\right), & \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \qquad \lambda = \lambda^*.$$

3. MIXED PROBLEM: $\alpha_+ = 1$, $\alpha_- = 0$; $\gamma_+ = 0$, $\gamma_- = 1$. Equation (1.8) takes the form

$$\beta\sin(\lambda\omega_0) + \lambda(a_+ + a_-)\cos(\lambda\omega_0) = \lambda(a_+ - a_-). \tag{1.9}$$

In particular, if $\beta = 0$, then

$$\lambda = \frac{2}{\omega_0}\arctan\sqrt{\frac{a_-}{a_+}} > 1, \qquad \text{if } \omega_0 < 2\arctan\sqrt{\frac{a_-}{a_+}}$$

as $a_+a_- > 0$; and the corresponding eigenfunction is

$$\psi(\omega) = \begin{cases} \cos(\lambda\omega) + \sqrt{\dfrac{a_-}{a_+}} \cdot \sin(\lambda\omega), & \omega \in (0, \frac{\omega_0}{2}); \\ \cos(\lambda\omega) + \sqrt{\dfrac{a_+}{a_-}} \cdot \sin(\lambda\omega), & \omega \in (-\frac{\omega_0}{2}, 0). \end{cases}$$

If λ is the least positive root of the transcendental equation (1.9), then we find the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} \sin\dfrac{\lambda\omega_0}{2}\cos\lambda\left(\omega - \dfrac{\omega_0}{2}\right), & \omega \in (0, \frac{\omega_0}{2}); \\ \cos\dfrac{\lambda\omega_0}{2}\sin\lambda\left(\omega + \dfrac{\omega_0}{2}\right), & \omega \in (-\frac{\omega_0}{2}, 0). \end{cases}$$

4. THE ROBIN PROBLEM: $\alpha_\pm = 1$; $\gamma_\pm \neq 0$. Equation (1.8) takes the form

$$\begin{aligned} &\beta(\lambda^2 a_+ a_- + \gamma_+\gamma_-) + \lambda^2(a_+ - a_-)(a_-\gamma_+ - a_+\gamma_-) \\ &\qquad + \lambda[\beta(a_-\gamma_+ + a_+\gamma_-) + (a_+ + a_-)(\gamma_+\gamma_- - \lambda^2 a_+ a_-)]\sin(\lambda\omega_0) \\ &\qquad + [\beta(\lambda^2 a_+ a_- - \gamma_+\gamma_-) + \lambda^2(a_+ + a_-)(a_-\gamma_+ + a_+\gamma_-)]\cos(\lambda\omega_0) \\ &\quad = 0. \end{aligned}$$

In particular, in the case of the problem without the interface ($a_+ = a_- = 1$, $\beta = 0$) we obtain the least eigenvalue as the least positive root of the transcendental equation

$$\tan(\lambda\omega_0) = \frac{\lambda(\gamma_+ + \gamma_-)}{\lambda^2 - \gamma_+\gamma_-} \tag{1.10}$$

and the corresponding eigenfunction is

$$\psi(\omega) = \lambda\cos\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right] - \gamma_+\sin\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right]$$

(see [3], §10.1.7).

In order to have $\lambda > 1$ we show that the condition $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2}$ from the assumption (b) of our Theorem is satisfied. In fact, we rewrite the equation (1.10) in the equivalent form $\lambda = \frac{1}{\omega_0}\left(\arctan\frac{\gamma_+}{\lambda} + \arctan\frac{\gamma_-}{\lambda}\right)$. It follows that

$$1 < \lambda < \frac{1}{\omega_0}(\arctan\gamma_+ + \arctan\gamma_-) \Longrightarrow \omega_0 < \arctan\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-}, \quad \text{provided that } \gamma_+\gamma_- < 1 \tag{1.11}$$

has to be fulfilled. But our condition from the assumption (b) means that $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2}$. Hence we obtain

$$\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-} \geq \frac{2\gamma_0}{1 - \gamma_0^2} > \frac{2\tan\frac{\omega_0}{2}}{1 - \tan^2\frac{\omega_0}{2}} = \tan\omega_0, \qquad \omega_0 < \frac{\pi}{2}.$$

Thus we established (1.11). In the case $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2} \geq 1$ for $\omega_0 \in [\frac{\pi}{2}, \pi)$ the inequality $\lambda > 1$ is fulfilled a fortiori, because of the property of the monotonic increase of the eigenvalues together with the increase of $\gamma(\omega)$ (see for example [4], chapter VI, §2, Theorem 6). In fact, $\lambda = 1$ is the solution of the equation (1.10) under assumption $\gamma_\pm = \tan\frac{\omega_0}{2}$.

2. Problem (L)

We assume that $G = G_+ \cup G_- \cup \Sigma_0$ is divided into two subdomains G_+ and G_- by a hyperplane $\Sigma_0 = G \cap \{x_n = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$. We assume also that $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known and, without loss of generality, that there exists $d > 0$ such that G_0^d is *a rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0, 2\pi)$, thus

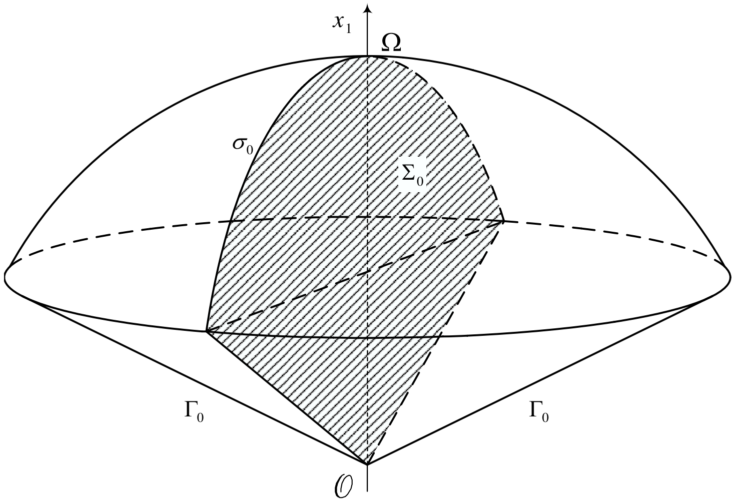

Fig. 3

$$\Gamma_0^d = \left\{ (r, \omega) \,\middle|\, x_1^2 = \cot^2\frac{\omega_0}{2} \sum_{i=2}^{n} x_i^2\,;\ r \in (0, d),\ \omega_1 = \frac{\omega_0}{2} \right\}.$$

Definition 2

A function $u(x)$ is called a *weak* solution of the problem (L) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G)$ and satisfies the integral identity

$$\int_G \left\{ a^{ij}(x) u_{x_j} \eta_{x_i} - a^i(x) u_{x_i} \eta(x) - a(x) u \eta(x) \right\} dx$$

$$+\int\limits_{\Sigma_0} \frac{\beta(\omega)}{r} u(x)\eta(x)\,ds + \int\limits_{\partial G} \frac{\gamma(\omega)}{r} u(x)\eta(x)\,ds$$
$$= \int\limits_{\partial G} g(x)\eta(x)\,ds + \int\limits_{\Sigma_0} h(x)\eta(x)\,ds - \int\limits_{G} f(x)\eta(x)\,dx$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_0$.

Regarding the equation we assume that the following conditions are satisfied:

(a) *the condition of the uniform ellipticity:*

$$\nu_\pm \xi^2 \le a^{ij}_\pm(x)\xi_i\xi_j \le \mu_\pm \xi^2, \quad \forall x \in \overline{G_\pm}, \quad \forall \xi \in \mathbb{R}^n;$$
$$\nu_\pm, \mu_\pm = \text{const} > 0, \ \text{and } a^{ij}_\pm(0) = a\delta^j_i\,,$$

where δ^j_i *is the Kronecker symbol and*

$$a = \begin{cases} a_+\,, & x \in G_+\,; \\ a_-\,, & x \in G_-\,, \end{cases}$$

with positive constants $a_\pm$*; we denote*

$$a_* = \min\{a_+, a_-\} > 0, \qquad a^* = \max\{a_+, a_-\} > 0;$$
$$\nu_* = \min\{\nu_-, \nu_+\}; \qquad \mu^* = \max(\mu_-, \mu_+);$$

(b) $a^{ij}(x) \in \mathbf{C}^0(\overline{G})$, $a^i(x) \in \mathbf{L}_p(G)$, $a(x), f(x) \in \mathbf{L}_{\frac{p}{2}}(G) \cap \mathbf{L}_2(G)$; $p > n$. *The inequalities*

$$\left(\sum_{i,j=1}^n |a^{ij}_\pm(x) - a^{ij}_\pm(y)|^2\right)^{\frac{1}{2}} \le a_\pm \mathcal{A}(|x-y|);$$
$$|x|\left(\sum_{i=1}^n |a^i_\pm(x)|^2\right)^{\frac{1}{2}} + |x|^2|a_\pm(x)| \le a_\pm \mathcal{A}(|x|)$$

hold for $x, y \in \overline{G}$, *Here* $\mathcal{A}(r)$ *is a monotonically increasing, nonnegative function, continuous at* 0 *with* $\mathcal{A}(0) = 0$*;*

(c) $a(x) \le 0$ *in* G*;* $\beta(\omega) \ge \nu_0 > 0$ *on* σ_0*;* $\gamma(\omega) \ge \nu_0 > 0$ *on* ∂G*;*

(d) *there exist numbers* $f_1 \ge 0$, $g_1 \ge 0$, $h_1 \ge 0$, $s > 1$, $\beta \ge s-2$ *such that*

$$|f(x)| \le f_1|x|^\beta, \qquad |g(x)| \le g_1|x|^{s-1}, \qquad |h(x)| \le h_1|x|^{s-1},$$

$\gamma(\omega)$ *is a positive bounded piecewise smooth function on* $\partial\Omega$, $\sigma(\omega)$ *is a positive continuous function on* σ_0*;*

(aa) $\left|\sum\limits_{i=1}^{n} \frac{\partial a^{ij}(x)}{\partial x_i}\right| \leq K$ *for all* $j = 1, \ldots, n$.

Our main result is the following theorem.

THEOREM 2
Let u *be a weak solution of the problem* (L), *the assumptions* (a)-(d), (aa) *are satisfied with* $\mathcal{A}(r)$ *Dini-continuous at zero. Let* λ *be as in* (1.1); $N = 2$. *Then there are* $d \in (0,1)$ *and constants* $C > 0$, $c > 0$ *depending only on* n, ν_*, μ^*, p, λ, $\left\|\sum_{i=1}^{n} |a^i(x)|^2\right\|_{\mathbf{L}_{\frac{p}{2}}(G)}$, K, ω_0, f_1, h_1, g_1, ν_0, s, M_0, $\operatorname{meas} G$, $\operatorname{diam} G$ *and on the quantity* $\int_0^1 \frac{\mathcal{A}(r)}{r}\,dr$ *such that for all* $x \in G_0^d$

$$|u(x)| \leq C_0 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1 + \frac{1}{\sqrt{\sigma_0}} h_1 \right) \cdot \begin{cases} |x|^{\lambda}, & \text{if } s > \lambda; \\ |x|^{\lambda} \ln^c \left(\frac{1}{|x|} \right), & \text{if } s = \lambda; \\ |x|^{s}, & \text{if } s < \lambda. \end{cases}$$

Suppose, in addition, that

$$a^{ij}(x) \in \mathbf{C}^1(G), \qquad \sigma(\omega) \in C^1(\sigma_0), \qquad \gamma(\omega) \in \mathbf{C}^1(\partial G),$$

$$f(x) \in \mathbf{V}^0_{p,2p-n}(G), \qquad h(x) \in V^{1-\frac{1}{p}}_{p,2p-n}(\sigma_0), \qquad g(x) \in \mathbf{V}^{1-\frac{1}{p}}_{p,2p-n}(\partial G);$$

$p > n$ *and there is a number*

$$\tau_s =: \sup_{\varrho > 0} \varrho^{-s} \Big(\|h\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Sigma^{\varrho}_{\varrho/2})} + \|g\|_{\mathbf{V}^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^{\varrho}_{\varrho/2})} \Big).$$

Then for all $x \in G_0^d$

$$|\nabla u(x)| \leq C_1 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1 + \frac{1}{\sqrt{\sigma_0}} h_1 + \tau_s \right) \times \begin{cases} |x|^{\lambda-1}, & \text{if } s > \lambda; \\ |x|^{\lambda-1} \ln^c \left(\frac{1}{|x|} \right), & \text{if } s = \lambda; \\ |x|^{s-1}, & \text{if } s < \lambda. \end{cases}$$

Furthermore, $u \in \mathbf{V}^2_{p,2p-n}(G)$ *and*

$$\|u\|_{\mathbf{V}^2_{p,2p-n}(G_0^{\varrho})} \leq C_2 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1 + \frac{1}{\sqrt{\sigma_0}} h_1 + \tau_s \right) \times \begin{cases} \varrho^{\lambda}, & \text{if } s > \lambda; \\ \varrho^{\lambda} \ln^c \left(\frac{1}{\varrho} \right), & \text{if } s = \lambda; \\ \varrho^{s}, & \text{if } s < \lambda. \end{cases}$$

3. Problem (WL)

We consider problem (WL) that is the transmission problem for a quasi-linear equation with semi-linear principal part.

DEFINITION 3
The function $u(x)$ is called a *weak* solution of the problem (WL) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^1(G)$ and satisfies the integral identity

$$\int_G \left\{ |u|^q a^{ij}(x) u_{x_j} \eta_{x_i} + b(x,u,u_x)\eta(x) \right\} dx$$
$$+ \int_{\Sigma_0} \frac{\beta(\omega)}{r} u|u|^q \eta(x)\, ds + \int_{\partial G} \frac{\gamma(\omega)}{r} u|u|^q \eta(x)\, ds$$
$$= \int_{\partial G} g(x,u)\eta(x)\, ds + \int_{\Sigma_0} h(x,u)\eta(x)\, ds$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^1(G)$.

Regarding the equation we assume that the following conditions are satisfied.

Let $q \geq 0$, $0 \leq \mu < q+1$, $s > 1$, $f_1 \geq 0$, $g_1 \geq 0$, $h_1 \geq 0$, $\beta \geq s-2$ be given numbers.

(a) *The condition of the uniform ellipticity:*

$$a_\pm \xi^2 \leq a_\pm^{ij}(x)\xi_i\xi_j \leq A_\pm \xi^2, \quad \forall x \in \overline{G_\pm}, \quad \forall \xi \in \mathbb{R}^n;$$
$$a_\pm, A_\pm = const > 0,\ a_\pm^{ij}(0) = a\delta_i^j\,,$$

where δ_i^j is the Kronecker symbol;

$$a = \begin{cases} a_+\,, & x \in G_+\,; \\ a_-\,, & x \in G_-\,; \end{cases}$$

$$\begin{cases} a_* = \min\{a_+,\, a_-\} > 0; \\ a^* = \max\{a_+,\, a_-\} > 0; \\ A^* = \max(A_-, A_+). \end{cases}$$

(b) $a_\pm^{ij}(x) \in \mathbf{C}^0(\overline{G})$ *and the inequality*

$$\left(\sum_{i,j=1}^{n} |a_\pm^{ij}(x) - a_\pm^{ij}(y)|^2 \right)^{\frac{1}{2}} \leq \mathcal{A}(|x-y|)$$

holds for $x, y \in \overline{G}$, *where* $\mathcal{A}(r)$ *is a monotonically increasing, nonnegative function, continuous at* 0 *with* $\mathcal{A}(0) = 0$.

(c) $|b(x, u, u_x)| \leq a\mu|u|^{q-1}|\nabla u|^2 + b_0(x)$; $0 \leq \mu < 1 + q$, $b_0(x) \in L_{p/2}(G)$, $n < p < 2n$.

(d) $\beta(\omega) \geq \nu_0 > 0$ *on* σ_0; $\gamma(\omega) \geq \nu_0 > 0$ *on* ∂G.

(e) $\frac{\partial h(x,u)}{\partial u} \leq 0$, $\frac{\partial g(x,u)}{\partial u} \leq 0$.

(f) $|b_0(x)| \leq f_1|x|^{\beta}$, $|g(x, 0)| \leq g_1|x|^{s-1}$, $|h(x, 0)| \leq h_1|x|^{s-1}$.

We assume without loss of generality that there exists $d > 0$ such that G_0^d is *a rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0, 2\pi)$.

Our main result is the following statement.

THEOREM 3
Let u be a weak solution of the problem (WL), *the assumptions* (a)-(f) *are satisfied with* $\mathcal{A}(r)$ *Dini-continuous at zero. Let us assume that* $M_0 = \max_{x\in\overline{G}} |u(x)|$ *is known. Let* λ *be as in* (1.1) *for* $N = 2$. *Then there are* $d \in (0, 1)$ *and constants* $C_0 > 0$, $c > 0$ *depending only on* n, a_*, A^*, p, q, λ, μ, f_1, h_1, g_1, ν_0, s, M_0, $\operatorname{meas} G$, $\operatorname{diam} G$ *and on the quantity* $\int_0^1 \frac{\mathcal{A}(r)}{r}\,dr$ *such that for all* $x \in G_0^d$

$$|u(x)| \leq C_0 \begin{cases} |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}}, & \text{if } s > \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}} \ln^c\left(\dfrac{1}{|x|}\right), & \text{if } s = \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{s}{q+1}}, & \text{if } s < \lambda\dfrac{1+q-\mu}{1+q}. \end{cases}$$

Suppose, in addition, that coefficients of the problem (WL) *satisfy such conditions, which guarantee the local estimate* $|\nabla u|_{0,G'} \leq M_1$ *for any smooth* $G' \subset \overline{G} \setminus \{\mathcal{O}\}$ *(see for example [1], §4). Then for all* $x \in G_0^d$

$$|\nabla u(x)| \leq C_1 \begin{cases} |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}-1}, & \text{if } s > \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}-1} \ln^c\left(\dfrac{1}{|x|}\right), & \text{if } s = \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{s}{q+1}-1}, & \text{if } s < \lambda\dfrac{1+q-\mu}{1+q} \end{cases}$$

with $C_1 = c_1\big(\|u\|_{2(q+1),G} + f_1 + g_1 + h_1\big)$, *where* c_1 *depends on* M_0, M_1 *and* C_0 *from above.*

The idea of the proofs of Theorems 1-3 is based on the deduction of a new inequality of the Friedrichs–Wirtinger type with the *exact* constant as well as other integral-differential inequalities adapted to the transmission problem. The precise exponent of the solution decrease rate depends on this exact constant. We obtain *the Friedrichs–Wirtinger type inequality* by the variational principle:

LEMMA 1
Let ϑ be the smallest positive eigenvalue of the problem (EVP). Let $\Omega \subset S^{n-1}$ be a bounded domain. Let $\psi \in \mathbf{W}^1(\Omega)$ and satisfy the boundary and conjunction conditions from (EVP) in the weak sense. Let $\gamma(\omega)$ be a positive bounded piecewise smooth function on $\partial\Omega$, $\beta(\omega)$ be a positive continuous function on σ_0. Then

$$\vartheta \int_{\Omega} a\psi^2(\omega)\, d\Omega \leq \int_{\Omega} a|\nabla_\omega \psi(\omega)|^2\, d\Omega + \int_{\sigma_0} \beta(\omega)\psi^2(\omega)\, d\sigma + \int_{\partial\Omega} \alpha(x)\gamma(\omega)\psi^2(\omega)\, d\sigma.$$

LEMMA 2
Let G_0^d be the conical domain and $\nabla v(\varrho, \cdot) \in \mathbf{L}_2(\Omega)$ for almost all $\varrho \in (0,d)$. Assume that for almost all $\varrho \in (0,d)$

$$V(\rho) = \int_{G_0^\varrho} ar^{2-n}|\nabla v|^2\, dx + \int_{\Sigma_0^\varrho} r^{1-n}\beta(\omega)v^2(x)\, ds + \int_{\Gamma_0^\varrho} r^{1-n}\gamma(\omega)v^2(x)\, ds < \infty.$$

Then

$$\int_{\Omega} a\left(\varrho v\frac{\partial v}{\partial r} + \frac{n-2}{2}v^2\right)\Bigg|_{r=\varrho} d\Omega \leq \frac{\varrho}{2\lambda}V'(\varrho).$$

At last we derive a result that asserts *the local estimate at the boundary* (near the conical point) of the weak solution of problem (WL).

THEOREM 4
Let $u(x)$ be a weak solution of the problem (WL). Suppose that assumptions (a), (c)-(e) are satisfied. Suppose, in addition, that $h(x) \in L_\infty(\Sigma_0)$, $g(x) \in L_\infty(\partial G)$. Then the inequality

$$\sup_{G_0^{\varkappa\varrho}} |u(x)| \leq \frac{C}{(1-\varkappa)^{\frac{n}{t}(q+1)}}\Big\{\varrho^{-\frac{n}{t}(q+1)}\|u\|_{t(q+1),G_0^\varrho} + \varrho^{\frac{2}{q+1}(1-\frac{n}{p})}\|b_0\|_{p/2,G_0^\varrho}^{\frac{1}{q+1}} + \varrho^{\frac{1}{q+1}}\left(\|g(x,0)\|_{\infty,\Gamma_0^\varrho}^{\frac{1}{q+1}} + \|h(x,0)\|_{\infty,\Sigma_0^\varrho}^{\frac{1}{q+1}}\right)\Big\}$$

holds for any $t > 0$, $\varkappa \in (0,1)$ and $\varrho \in (0,d)$, where $C = \mathrm{const}(n, a_*, A^*, t, p, \mu, G)$ and $d \in (0,1)$.

Examples

Here we consider two dimensional transmission problem for the Laplace operator with absorbtion term in an angular domain and investigate the corresponding eigenvalue problem. Suppose $n = 2$, and the domain G lies inside the angle

$$G_0 = \Big\{(r,\omega)\ \Big|\ r > 0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}\Big\}, \quad \omega_0 \in]0, 2\pi[\,;$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_\pm = \Big\{(r,\omega)\ \Big|\ r > 0;\ \omega = \pm\frac{\omega_0}{2}\Big\}, \qquad \Sigma_0 = \{(r,\omega)\ |\ r > 0;\ \omega = 0\}$$

and we put

$$\beta(\omega)\big|_{\Sigma_0} = \beta = \text{const} \geq 0, \qquad \gamma(\omega)\big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = \text{const} > 0.$$

We consider the following problem:

$$\begin{cases} \dfrac{d}{dx_i}\left(|u|^q u_{x_i}\right) = a_0 r^{-2} u|u|^q - \mu u|u|^{q-2}|\nabla u|^2, & x \in G_0 \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0; & \\ \left[a|u|^q \dfrac{\partial u}{\partial n}\right]_{\Sigma_0} + \dfrac{\beta}{|x|} u|u|^q = 0, & x \in \Sigma_0; \\ \alpha_\pm a_\pm |u_\pm|^q \dfrac{\partial u_\pm}{\partial n} + \dfrac{\gamma_\pm}{|x|} u_\pm |u_\pm|^q = 0, & x \in \Gamma_\pm \setminus \mathcal{O}, \end{cases}$$

where

$$a = \begin{cases} a_+\,, & x \in G_+\,; \\ a_-\,, & x \in G_-\,, \end{cases}$$

$a_\pm$ are positive constants; $a_0 \geq 0$, $0 \leq \mu < 1 + q$, $q \geq 0$; $\alpha_\pm \in \{0; 1\}$. We make the function change $u = v|v|^{\varsigma-1}$ with $\varsigma = \frac{1}{q+1}$ and consider our problem for the function $v(x)$:

$$\begin{cases} \triangle v + \mu\varsigma v^{-1}|\nabla v|^2 = a_0(1+q)r^{-2}v, \quad \varsigma = \dfrac{1}{1+q}, & x \in G_0 \setminus \Sigma_0; \\ [v]_{\Sigma_0} = 0; & \\ \left[a\dfrac{\partial v}{\partial n}\right]_{\Sigma_0} + (1+q)\beta\dfrac{v(x)}{|x|} = 0, & x \in \Sigma_0; \\ \alpha_\pm a_\pm \dfrac{\partial v_\pm}{\partial n} + (1+q)\gamma_\pm \dfrac{v_\pm(x)}{|x|} = 0, & x \in \Gamma_\pm \setminus \mathcal{O}. \end{cases}$$

We want to find the exact solution of this problem in the form $v(r,\omega) = r^{\varkappa}\psi(\omega)$. For $\psi(\omega)$ we obtain the problem

$$\begin{cases} \psi''(\omega) + \dfrac{\mu\varsigma}{\psi(\omega)}\psi'^2(\omega) + \big\{(1+\mu\varsigma)\varkappa^2 - a_0(1+q)\big\}\cdot\psi(\omega) = 0, \\ \qquad\qquad\qquad\qquad \omega \in (-\frac{\omega_0}{2}, 0) \cup (0, \frac{\omega_0}{2}); \\ [\psi]_{\omega=0} = 0; \\ [a\psi'(0)] = (1+q)\beta\psi(0); \\ \pm\alpha_\pm a_\pm \psi'_\pm\left(\pm\dfrac{\omega_0}{2}\right) + (1+q)\gamma_\pm\psi_\pm\left(\pm\dfrac{\omega_0}{2}\right) = 0. \end{cases}$$

We assume that $\varkappa^2 > a_0\frac{(1+q)^2}{1+q+\mu}$ and define the value $\Upsilon = \sqrt{\varkappa^2 - a_0\frac{(1+q)^2}{1+q+\mu}}$. We consider separately two cases: $\mu = 0$ and $\mu \neq 0$.

The case $\mu = 0$

In this case we get

$$\psi_\pm(\omega) = A\cos(\Upsilon\omega) + B_\pm\sin(\Upsilon\omega),$$

where constants A, $B_\pm$ should be determined from the conjunction and boundary conditions.

1. THE DIRICHLET PROBLEM: $\alpha_\pm = 0$, $\gamma_\pm \neq 0$. Direct calculations will give

$$\psi_\pm(\omega) = \cos(\Upsilon\omega) \mp \cot\left(\Upsilon\frac{\omega_0}{2}\right)\cdot\sin(\Upsilon\omega), \qquad \Upsilon = \begin{cases} \dfrac{\pi}{\omega_0}, & \text{if } \beta = 0; \\ \Upsilon^*, & \text{if } \beta \neq 0, \end{cases}$$

where Υ^* is the least positive root of the transcendental equation

$$\Upsilon\cdot\cot\left(\Upsilon\frac{\omega_0}{2}\right) = -\frac{1+q}{a_+ + a_-}\beta$$

and from the graphic solution we obtain $\frac{\pi}{\omega_0} < \Upsilon^* < \frac{2\pi}{\omega_0}$. The corresponding eigenfunctions are

$$\psi_\pm(\omega) = \begin{cases} \cos\left(\dfrac{\pi\omega}{\omega_0}\right), & \text{if } \beta = 0; \\ \cos(\Upsilon^*\omega) \mp \cot\left(\Upsilon^*\frac{\omega_0}{2}\right)\cdot\sin(\Upsilon^*\omega), & \text{if } \beta \neq 0. \end{cases}$$

2. THE NEUMANN PROBLEM: $\alpha_\pm = 1$, $\gamma_\pm = 0$. Direct calculations give

$$\Upsilon = \begin{cases} \dfrac{\pi}{\omega_0}, & \text{if } \beta = 0; \\ \Upsilon^*, & \text{if } \beta \neq 0, \end{cases}$$

where Υ^* is the least positive root of the transcendental equation

$$\Upsilon \cdot \tan\left(\Upsilon \frac{\omega_0}{2}\right) = \frac{1+q}{a_+ + a_-}\beta$$

and from the graphic solution we obtain $0 < \Upsilon^* < \frac{\pi}{\omega_0}$. The corresponding eigenfunctions are

$$\psi_\pm(\omega) = \begin{cases} a_\mp \sin\left(\dfrac{\pi\omega}{\omega_0}\right), & \text{if } \beta = 0; \\ \cos(\Upsilon^*\omega) \pm \tan\left(\Upsilon^* \frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), & \text{if } \beta \neq 0. \end{cases}$$

3. MIXED PROBLEM: $\alpha_+ = 1$, $\alpha_- = 0$; $\gamma_+ = 0$, $\gamma_- = 1$. Direct calculations give: $\Upsilon = \Upsilon^*$, where Υ^* is the least positive root of the transcendental equation

$$a_+ \tan\left(\Upsilon \frac{\omega_0}{2}\right) - a_- \cot\left(\Upsilon \frac{\omega_0}{2}\right) = \frac{1+q}{\Upsilon}\beta.$$

The corresponding eigenfunctions are

$$\psi_+(\omega) = \cos(\Upsilon^*\omega) + \tan\left(\Upsilon^* \frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), \qquad \omega \in \left[0, \tfrac{\omega_0}{2}\right];$$
$$\psi_-(\omega) = \cos(\Upsilon^*\omega) + \cot\left(\Upsilon^* \frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), \qquad \omega \in \left[-\tfrac{\omega_0}{2}, 0\right].$$

4. THE ROBIN PROBLEM: $\alpha_\pm = 1$, $\gamma_\pm \neq 0$. Direct calculations give:

   1) $\frac{\gamma_+}{\gamma_-} = \frac{a_+}{a_-} \Longrightarrow \psi_\pm(\omega) = a_\mp \sin(\Upsilon^*\omega)$, where Υ^* is the least positive root of the transcendental equation

   $$\Upsilon \cdot \cot\left(\Upsilon \frac{\omega_0}{2}\right) = -(1+q)\frac{\gamma_+}{a_+}$$

   and from the graphic solution we obtain $\frac{\pi}{\omega_0} < \Upsilon^* < \frac{2\pi}{\omega_0}$.

   2) $\frac{\gamma_+}{\gamma_-} \neq \frac{a_+}{a_-} \Longrightarrow A \neq 0$ and $\psi_\pm(0) \neq 0$;

further see below the general case $\mu \neq 0$.

The case $\mu \neq 0$

It is obvious that in this case $\psi(0) \neq 0$. By setting $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$, we arrive at the problem for $y(\omega)$

$$\begin{cases} y' + (1+\mu\varsigma)y^2(\omega) + (1+\mu\varsigma)\varkappa^2 - a_0(1+q) = 0, \quad \omega \in \left(-\frac{\omega_0}{2}, 0\right) \cup \left(0, \frac{\omega_0}{2}\right); \\ a_+ y_+(0) - a_- y_-(0) = (1+q)\beta; \\ \pm\alpha_\pm a_\pm y_\pm\left(\pm\frac{\omega_0}{2}\right) + (1+q)\gamma_\pm = 0. \end{cases}$$

Integrating the equation of our problem we find

$$y_\pm(\omega) = \Upsilon \tan\left\{\Upsilon\left(C_\pm - (1+\mu\varsigma)\omega\right)\right\}, \qquad \forall C_\pm.$$

From the boundary conditions we have

$$C_\pm = \pm(1+\mu\varsigma)\frac{\omega_0}{2} \mp \frac{1}{\Upsilon}\arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}.$$

Finally, in virtue of the conjunction condition, we get the equation for $\varkappa$:

$$\begin{aligned} a_+ \cdot &\frac{\alpha_+ a_+ \Upsilon \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\} - (1+q)\gamma_+}{\alpha_+ a_+ \Upsilon + (1+q)\gamma_+ \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\}} \\ &\quad + a_- \cdot \frac{\alpha_- a_- \Upsilon \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\} - (1+q)\gamma_-}{\alpha_- a_- \Upsilon + (1+q)\gamma_- \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\}} \\ &= \frac{1+q}{\Upsilon}\beta, \qquad \text{where } 1+\mu\varsigma = \frac{1+q+\mu}{1+q}. \end{aligned}$$

Thus we obtain

$$y_\pm(\omega) = \Upsilon \tan\left\{\Upsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}\right\}$$

and, because of $(\ln\psi(\omega))' = y(\omega)$, it follows that

$$\psi_\pm(\omega) = \cos^{\frac{1+q}{1+q+\mu}}\left\{\Upsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}\right\}.$$

At last, returning to the function u we establish a solution of our problem

$$u_\pm(r,\omega) = r^{\frac{\varkappa}{1+q}}\cos^{\frac{1}{1+q+\mu}}\left\{\Upsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}\right\}.$$

If we consider **the Dirichlet problem without the interface**: $\alpha_\pm = 0$, $a_\pm = 1$, $\beta = 0$, then we can calculate

$$u(r,\omega) = r^{\widetilde{\lambda}}\cos^{\frac{1}{1+q+\mu}}\left(\frac{\pi\omega}{\omega_0}\right); \qquad \widetilde{\lambda} = \frac{\sqrt{(\pi/\omega_0)^2 + a_0(1+q+\mu)}}{1+q+\mu}.$$

It recovers a well known result (see [2], p. 374, Example 4.6). Now we can verify that the derived exact solution satisfies the estimate of Theorem 3. In fact, in our case we have: the value λ is equal $\vartheta = \frac{\pi}{\omega_0}$ and therefore

$$|u(r,\omega)| \le r^{\widetilde{\lambda}} \le r^{\frac{\pi}{\omega_0}\cdot\frac{1}{1+q+\mu}} \le r^{\frac{\pi}{\omega_0}\cdot\frac{1+q-\mu}{(1+q)^2}},$$

since $a_0 \ge 0$ and $\frac{1}{1+q+\mu} \ge \frac{1+q-\mu}{(1+q)^2}$.

References

[1] M.V. Borsuk, *A priori estimates and solvability of second order quasilinear elliptic equations in a composite domain with nonlinear boundary conditions and conjunction condition*, Proc. Steklov Inst. of Math. **103** (1970), 13-51.

[2] M. Borsuk, *The behavior of weak solutions to the boundary value problems for elliptic quasilinear equations with triple degeneration in a neighborhood of a boundary edge*, Nonlinear Anal. **56** (2004), no 3, 347-384.

[3] M. Borsuk, V. Kondratiev, *Elliptic boundary value problems of second order in piecewise smooth domains*, North-Holland Mathematical Library **69**, Elsevier Science B.V., Amsterdam, 2006.

[4] R. Courant, D. Hilbert, *Methoden der mathematischen Physik, Bd. 1*, Springer-Verlag, Berlin, 1931.

Departament of Mathematics and Informatics
University of Warmia and Mazury in Olsztyn
10-957 Olsztyn–Kortowo
E-mail: borsuk@uwm.edu.pl

Received: 7 May 2008; final version: 31 May 2008;
available online: 29 September 2008.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Svetlana Lebed, Sergei Rogosin

Optimal design of unbounded 2D composite materials with circular inclusions of different radii

Abstract. Optimal design problem for 2D composite materials with different circular inclusions is studied on the base of the potential method combined with functional equation method. Exact geometric description of the optimal distribution of the inclusions is determined.

1. Introduction

The article is devoted to the constructive analysis of mathematical models arising at the study of optimal design of 2D composite materials (see e.g. [5]).

The optimal design problem in the considered case is the problem of the determination of a distribution of circular inclusions in a matrix of homogeneous material in such a way that the obtained inhomogeneous material possesses an extremal (minimal or maximal) effective conductivity in a given direction.

Potential analysis is used in combination with the method of functional equations (for wider description of the approach see [6] and [7]). Such approach makes possible to discover general properties of composite materials on the base of an explicit representation of the effective conductivity functional. Besides, in certain special cases we have found an exact geometric description of an optimal (in the above sense) distribution of inclusions.

The paper continues the authors' study of behind optimal design problems which were previously devoted to the case of 2D composite materials with equal circular inclusions (see [5]). In particular, in [5] we studied the problem of optimal design of 2D unbounded composite materials in the case of small Bergmann parameter. The corresponding boundary conditions are simplified, namely only their main parts are considered. Such model situation allows us to give a complete geometric description of the optimal distribution of circular inclusions of equal radius. It was shown that the solution of the simplified boundary value problems gives the minimal or maximal value to the functional of the effective conductivity if each inclusion touches at least one of others. For small number of inclusions an exact description of the optimal distribution of

AMS (2000) Subject Classification: 30E25, 74E30, 74Q15, 80A20.

inclusions is given and exact optimal value of the changing part of the functional of the effective conductivity is calculated. Such model problem can be used for an approximation of the optimal design problem with sufficiently small concentration of inclusions ν.

Here we concern with the case of non-equal circular inclusions. We use the same argument. It should be noted that the constructive approach applied here differs from the recently studied models of optimal design based mainly on the homogenization technique (see e.g. [2], [1]).

2. A model

Let us consider 2D unbounded composite materials with circular inclusions of different radii. Let the matrix of a composite be geometrically modelled by an unbounded multiply connected circular domain, namely, an exterior of finite number of discs of different radii. These discs correspond to inclusions for which the radii are given but the position on the complex plane are subject of further determination. We suppose that the matrix is filled in by the homogeneous material of a constant (thermal) conductivity $\lambda_m = 1$, and the inclusions are filled in by another material of a constant conductivity $\lambda_i = \lambda$. We suppose additionally that the Bergmann parameter $\rho = \frac{\lambda-1}{\lambda+1}$ is sufficiently small ($|\rho| \ll 1$). The composite material is placed into the steady (thermal) field. In order to avoid indeterminancies, we consider only the case of positive Bergmann parameter, i.e., the conductivity of inclusions is greater than the conductivity of matrix.

The question is to determine the distribution of the inclusions for which the considered inhomogeneous composite material possesses an extremal (minimal or maximal) effective conductivity in a given direction (say in the direction of the positive real line). In the case of a small Bergmann parameter we use the same approach as in [5], namely, we simplify the boundary conditions by considering only the main part of them (with respect to the power of ρ) and then minimize or maximize the only changing part of the functional of the effective conductivity. Such simplification gives us possibility to obtain an analytic solution to the model problem. The later can be considered as an approximation to the starting optimal design problem. The model problem is studied by the reduction to a finite-dimensional extremal problem with centers of inclusions as unknown variables.

We have to note that our approach does not depend on the type of the considered physical field neither on the direction in which we determine the effective conductivity.

3. Solution to the optimal design problem in the case of a small Bergmann parameter

Let $D_k := \{z \in \mathbb{C} : \ |z - a_k| < r_k\}$, $k = 1, \dots, n$, $|a_j - a_k| \geq r_j + r_k$ and $a_j \neq a_k$, for $k \neq j$, be a finite number of disjoint discs and $L_k := \{z \in \mathbb{C} : \ |z - a_k| = r_k\}$ be their boundary circles. We consider an optimal design problem in the potential case, i.e., there are thermal potentials u_k in each disc D_k, $k = 1, \dots, n$, as well as a potential u in the domain $D = \mathbb{C} \setminus \bigcup_{k=1}^{n} \overline{D_k}$. We suppose that these potentials satisfy the ideal contact conditions on the boundary of inclusions $L = \bigcup_{k=1}^{n} L_k$. By introducing the complex potentials

$$\begin{aligned} \varphi(z) &= u(z) + iv(z), && z \in D, \\ \varphi_k(z) &= u_k(z) + iv_k(z), && z \in D_k, \ k = 1, \dots, n, \end{aligned} \tag{1}$$

we arrive at the $\mathbb{R}$-linear boundary value conditions on each circle L_k

$$\varphi(t) = \varphi_k(t) - \rho\overline{\varphi_k(t)} + g(t), \tag{2}$$

where $g(z)$ is a given function representing an external thermal field. It is well-known (see e.g. [6], [7]) that for a general domain problem, (2) does not admit an analytic solution. In the case of a multiply connected circular domain an analytic solution to the $\mathbb{R}$-linear boundary value problem with constant coefficients is obtained (see e.g. [7]) in the form of series with summations depending on behind certain group of symmetries. Anyway, even in this case we cannot use such representation in order to get an exact description of the optimal distribution of (circular) inclusions. That is why certain simplification of the problem is made.

Here we consider the case of unbounded composite materials with finite number of circular inclusions and with conductivities of matrix and inclusions close to each other (i.e., with small Bergmann parameter). Thus the concentration ν is equal to 0 and the second term in the right hand-side of (2) is sufficiently small. Thus we replace the starting optimal design problem by model one. The later consists in optimization of the changing part of the standard functional of the effective conductivity

$$\frac{\lambda_e}{\lambda_m} = 1 + \frac{2\nu\rho}{n} \sum_{k=1}^{n} \int_{L_k} \operatorname{Re} \varphi_k^-(t)\, dy, \qquad t = x + iy, \tag{3}$$

on the set of solutions to a simplified boundary value problem (which depend on position of the inclusions in the composite).

Therefore, our model optimal design problem is considered in the form: to find positions of the discs D_k, $k = 1, \dots, n$, such that the following functional σ possesses an optimal value, namely

$$\sigma := \sum_{k=1}^{n} \int_{L_k} \operatorname{Re} \varphi_k^-(t)\, dy \;\longrightarrow\; \min(\max), \tag{4}$$

under constrains (boundary conditions)

$$\varphi^+(t) - \varphi_k^-(t) = g(t), \qquad t \in L = \bigcup_{k=1}^{n} L_k\,, \tag{5}$$

where $t = x + iy$.

It is known (see [3]) that the solution of the problem (5) can be represented in the following form

$$\varphi^{\pm}(z) = \frac{1}{2\pi i} \int_L \frac{g(t)\, dt}{t - z}, \qquad z \in D^{\pm}, \tag{6}$$

where each of the circles L_k is clock-wise oriented. Thus the extremal distribution of domains D_k is determined by their centers and also by the values of a given function g. The determination of the extremal value of the functional (4) is in general rather complicated problem. The complete (and exact) solution is possible only in the case when the function g is explicitly given. Here we confine ourself to the important for mechanics of composite materials case $g(z) = \overline{z}$. In this case, the integrals (6) are calculated explicitly and the general solution to the problem (5) has the following form

$$\varphi(z) = \begin{cases} -\displaystyle\sum_{m=1}^{n} \frac{r_m^2}{z - a_m}, & z \in D^-, \\ \overline{a_k} - \displaystyle\sum_{m \neq k}^{n} \frac{r_m^2}{z - a_m}, & z \in D_k\,. \end{cases} \tag{7}$$

We calculate the value of the functional (4) by using the mean value theorem for harmonic functions (see [4])

$$\sigma = \int_L \operatorname{Re} \varphi^-(t)\, dy = \sum_{k=1}^{n} \int_{L_k} \operatorname{Re} \varphi^-(t)\, dy = \pi \sum_{k=1}^{n} r_k^2 \operatorname{Re} \left(\varphi^-\right)'(a_k). \tag{8}$$

Let us find the values $\left(\varphi_k^-\right)'(a_k)$:

$$\left(\varphi_k^-\right)'(a_k) = \sum_{m \neq k} \frac{r_m^2}{(z - a_m)^2} \bigg|_{z = a_k} = \sum_{m \neq k} \frac{r_m^2}{(a_k - a_m)^2}\,.$$

Substituting these values into (8), we get $\sigma = \operatorname{Re} \mu$, where

$$\mu = \pi \sum_{k=1}^{n} \sum_{m \neq k} \frac{r_m^2 r_k^2}{(a_k - a_m)^2} \, . \tag{9}$$

Therefore, the analysis of the initial extremal problem is reduced now to the study of the complex valued function μ of n complex variables $a_1, a_2, \ldots, a_n$. Since the value of the function μ is independent on the translation, we can fix one of the points a_k.

Lemma

Assume that the function $\sigma = \operatorname{Re}\mu$ attains its maximum on the set of points $A := \{a_1, a_2, \ldots, a_n\}$. Then

1) $\mu(A) \in \mathbb{R}$;

2) *each disc D_k is touched by at least one of other discs D_m, so that the closure of the domain $\overline{D^+}$ is a connected set on the complex plane $\mathbb{C}$.*

Proof. 1) Let us represent the functional μ given by (9) in the form $\mu = \mu_1 + i\mu_2$. Denote by $\mu(A)$ the value of the functional μ corresponding to the set of points A.

Consider the value of the function μ after rotation of the plane by a certain angle θ, i.e., corresponding to the set of points $A' = \{e^{i\theta}a_1, e^{i\theta}a_2, \ldots, e^{i\theta}a_n\}$:

$$\begin{aligned}\mu(A') &= \pi \sum_{k=1}^{n} \sum_{m \neq k} \frac{r_m^2 r_k^2}{e^{2i\theta}(a_k - a_m)^2} = e^{-2i\theta}(\mu_1(A) + i\mu_2(A)) \\ &= \mu_1(A)\cos 2\theta + \mu_2(A)\sin 2\theta + i(\mu_2(A)\cos 2\theta - \mu_1(A)\sin 2\theta).\end{aligned}$$

Therefore, $\sigma = \operatorname{Re}\mu = \mu_1 \cos 2\theta + \mu_2 \sin 2\theta$. One can choose the value of θ in such a way that

$$\mu(A') = |\mu(A)| = \sigma(A') \leq \sigma(A) = \operatorname{Re}\mu(A).$$

Hence $\operatorname{Im}\mu(A) = 0$ [1].

It follows that on the extremal set of points A the function

$$\mu(A) = \mu_1(A) = \pi \sum_{k=1}^{n} \sum_{m \neq k} \frac{r_m^2 r_k^2}{(a_k - a_m)^2}$$

has a real value.

2) Let $A := \{a_1, a_2, \ldots, a_n\}$ be an optimal set of centers, i.e., the functional σ attains its maximal value on A. Consider the corresponding optimal set of

[1]The authors are thankful to the referee who shows a shorter proof of the first assertion to Lemma.

discs with the centers $a_1, a_2, \ldots, a_n$ (for simplicity we can denote this set by A too). Introduce the following function

$$u(z) := \operatorname{Re} \sum_{m=2}^{n} \frac{r_m^2 r_1^2}{(z - a_k)^2} .$$

The sum $u(a_1)$ is the part of the sum σ, which is changing when the disc D_1 is moving. Under assumption, the function $u(z)$ attains its maximal value for $z = a_1$. Moreover, $|a_1 - a_k| > r_1 + r_k$, $k = 2, 3, \ldots, n$. But the function $u(z)$ is harmonic in the (in general, multiply connected) domain

$$\{z : \ |z - a_k| > r_1 + r_k, \ k = 2, 3, \ldots, n\}$$

and continuous in

$$\{z : \ |z - a_k| \geq r_1 + r_k, \ k = 2, 3, \ldots, n\},$$

vanishing at infinity. Therefore, the maximal value of the function $u(z)$ is attained at a boundary point of the above domain, i.e., when $|a_1 - a_k| = r_1 + r_k$ for certain k. Hence the optimal discs are touching each other.

Let us show now that the closure of the discs corresponding to the set A is a connected set. If not, then $A = A_1 \cup A_2$, with none disc from A_1 touching any disc from A_2. Let us fix one of the centers $a_p \in A_1$, and one of the centers $a_q \in A_2$. Represent another centers $a_k \in A_1$ in the form $a_k = a_p + b_{kp}$, and the centers $a_m \in A_2$ in the form $a_m = a_q + b_{qm}$, respectively. We consider the following function

$$u(z) := \operatorname{Re} \sum_{k,m} \frac{r_m^2 r_k^2}{(a_p - z + b_{kp} - b_{mq})^2} ,$$

where k, m are those values of indices for which $a_k \in A_1$, $a_m \in A_2$. The sum $u(a_q)$ is a part of the sum σ, which is changing when the mutual position of the sets A_1, A_2 is changing for fixed elements inside these sets. The variable z is modelling such changing. This variable is running along a compact set K in $\hat{\mathbb{C}}$, which described all possible changing of the mutual position of discs corresponding to A_1, A_2, up to the touching of these sets. The function $u(z)$ is harmonic in $\operatorname{int} K$ and continuous in K. By the Maximum Principle for harmonic functions this function has to attain its maximum on the boundary of K. The latter corresponds to the touching of certain discs from A_1 with certain discs from A_2. It contradicts our assumption and the Lemma is proved.

It follows from the Lemma that the optimal distribution of the discs always corresponds to the percolation situation, i.e., to the case when the discs are touching and constitute a connected set.

4. Exact geometric description of the solution in certain special cases

Let $n = 2$, $D_1 = \{z \in \mathbb{C} : |z - a_1| < r\}$, $D_2 = \{z \in \mathbb{C} : |z - a_2| < R\}$, $r < R$. Then $\mu_1 = \frac{2\pi r^2 R^2}{(a_1 - a_2)^2}$. Hence, $\mathrm{Im}\frac{1}{(a_1-a_2)^2} = 0$. It is possible in the following two cases:

a) $a_1 - a_2 = -(r + R)i$ (see Fig. 1.1), or $a_1 - a_2 = (r + R)i$ (see Fig. 1.2). In this case $\mu_1 = -\frac{2\pi r^2 R^2}{(R+r)^2}$. Therefore, the minimal value of the functional σ (which is the changing part of the effective conductivity functional) for composites with two circular inclusions is attained when the centers of these inclusions lay on the straight line parallel to imaginary axes, and these discs are touching each other, of course.

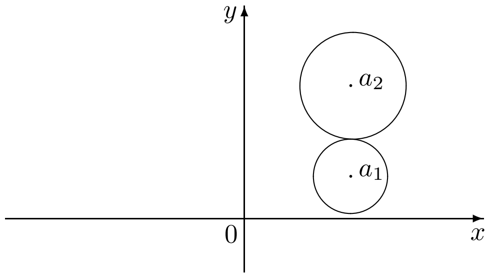

Fig. 1.1. Position of two inclusions corresponding to the minimal value of the functional (4)

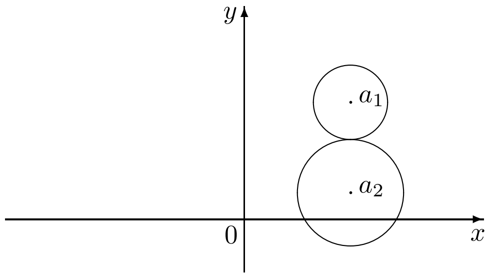

Fig. 1.2. Position of two inclusions corresponding to the minimal value of the functional (4)

b) $a_1 - a_2 = -(R + r)$ (see Fig. 1.3), or $a_1 - a_2 = R + r$ (see Fig. 1.4), and thus $\mu_1 = \frac{2\pi r^2 R^2}{(R+r)^2}$. Maximal value of the effective conductivity functional corresponds to the horizontal position of inclusions.

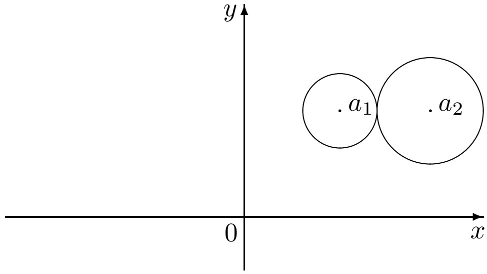

Fig. 1.3. Position of two inclusions corresponding to the maximal value of the functional (4)

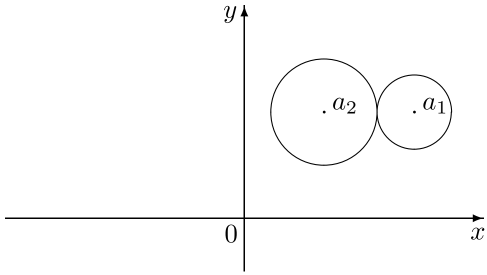

Fig. 1.4. Position of two inclusions corresponding to the maximal value of the functional (4)

EXAMPLE
In the case of three inclusions an optimal configuration is of the cluster type, i.e., three inclusions are touching each other.

For instance, let us consider three discs of radii $r = 1, 2, 4$, respectively. Put an origin of the coordinate system at the focal point of the triangle with edges at the centers of the touching discs. Then the value of the functional μ is equal to

$$\mu = -\frac{28\pi}{r^2} e^{i(\frac{\pi}{3} - 2\alpha)},$$

where $r = \frac{45}{4\sqrt{14}}$ is the circumradius of the triangle, and α is a rotation angle. The minimal (maximal) value of this functional is achieved at $\alpha = \frac{\pi}{3}$ ($\alpha = \frac{4}{3}\pi$), and is equal to $\mu = -\frac{6272}{2025}\pi$ ($\mu = +\frac{6272}{2025}\pi$, respectively).

For comparison, if we consider the chain of inclusions of the same radii, then minimal (maximal) value of μ is equal to $\mu = -\frac{80}{81}\pi$ ($\mu = +\frac{80}{81}\pi$, respectively).

Acknowledgement.

The work is partially supported by the Belarusian Fund for Fundamental Scientific Research. It is done in the framework of an Agreement on the Scientific Cooperation between Pedagogical University of Cracow and Belarusian State University.

The authors are grateful to an anonymous referee for careful reading of the manuscript and making several important remarks to improve its content.

References

[1] M.P. Bendsøe, *Optimization of structural topology, shape, and material*, Springer-Verlag, Berlin, 1995.

[2] A. Cherkaev, *Variational methods for structural optimization*, Applied Mathematical Sciences, **140**, Springer-Verlag, New York, 2000.

[3] F.D. Gakhov, *Краевые задачи* (Russian) [Boundary value problems], third edition, revised and augmented, "Nauka", Moscow, 1977.

[4] M.A. Lavrentiev, B.V. Shabat, *Методы теории функций комплексного переменного* (Russian) [Methods of the theory of functions of complex variables], "Nauka", Moscow, 1973.

[5] S.F. Makaruk, V.V. Mityushev, S.V. Rogosin, *An optimal design problem for two-dimensional composite materials. A constructive approach*, in: (A.A. Kilbas, S.V. Rogosin eds.) Analytic methods of analysis and differential equations: AMADE 2003, Cambridge Scientific Publishers Ltd, 2006, 155-170.

[6] V.V. Mityushev, E. Pesetskaya, S.V. Rogosin, *Analytical methods for heat conduction in composites and porous media*; in: (A. Öchsner, G.E. Murch, M.J.S de Lemos eds.) Cellular and porous materials: thermal properties simulation and prediction, Wiley-VCH, Weinheim, 2008, 121-164.

[7] V.V. Mityushev, S.V. Rogosin, *Constructive methods for linear and nonlinear boundary value problems for analytic functions. Theory and applications*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **108**, Chapman & Hall/CRC, Boca Raton, FL, 2000.

Brest Technical State University
Moskovskaya str. 267
224017 Brest
Belarus
E-mail: sfmakaruk@tut.by

Belarusian State University
Nezavisimosti ave 4
220030 Minsk
Belarus
E-mail: rogosin@gmail.com

Received: 7 May 2008; final version: 5 August 2008;
available online: 14 October 2008.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Moonja Jeong

Bergman kernel functions for planar domains and conformal equivalence of domains

Abstract. The Bergman kernels of multiply connected domains are related with proper holomorphic maps onto the unit disc. We study multiply connected planar domains and represent conformal equivalence of the Bell representative domains with annuli or any doubly connected domains by explicit formulae. We study the expression for the Bergman kernels of circular multiply connected planar domains.

1. Introduction

In this paper, we study the Bergman kernels of multiply connected domains and their Bell representations and circular multiply connected domains.

Let Ω be a bounded domain in $\mathbb{C}$. The Bergman projection P is the orthogonal projection of $L^2(\Omega)$ onto its subspace $H^2(\Omega)$ of holomorphic functions. The Bergman kernel $K_\Omega(\cdot,\cdot)$ is the kernel for P in the sense that for $f \in L^2(\Omega)$

$$Pf(z) = \int_\Omega K_\Omega(z,\zeta) f(\zeta)\, dA, \qquad z \in \Omega.$$

Let U be the unit disc in $\mathbb{C}$ with the area measure $dA = dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}$. Then the Bergman kernel for U is given by

$$K_U(z,\zeta) = \frac{1}{\pi} \frac{1}{\left(1 - z\overline{\zeta}\right)^2}, \qquad z,\zeta \in U. \tag{1.1}$$

Let $\Omega \neq \mathbb{C}$ be a simply connected planar domain and $f\colon \Omega \longrightarrow U$ be the Riemann map with $f(a) = 0$ and $f'(a) > 0$. The transformation formula for the Bergman kernel is

$$K_\Omega(z,\zeta) = f'(z) K_U(f(z), f(\zeta)) \overline{f'(\zeta)}. \tag{1.2}$$

AMS (2000) Subject Classification: Primary 30C20, 30C40, Secondary 30C35, 30E25.

It implies that

$$K_\Omega(z,\zeta) = \frac{1}{\pi}\frac{f'(z)\overline{f'(\zeta)}}{\left(1 - f(z)\overline{f(\zeta)}\right)^2} \qquad z,\zeta \in \Omega. \tag{1.3}$$

Hence, $K_\Omega(a,a) = \frac{1}{\pi}f'(a)^2$. Therefore, the derivative of $f(z)$ is determined through the Bergman kernel by the formula

$$f'(z) = K_\Omega(z,a)\sqrt{\frac{\pi}{K_\Omega(a,a)}}. \tag{1.4}$$

The transformation formula (1.2) for the Bergman kernel holds under any biholomorphic map between two domains. Let us determine the Bergman kernel for $\{z \in \mathbb{C} : |z| < 2\}$.

Example 1.1
Let $U' = \{z \in \mathbb{C} : |z| < 2\}$. Let $f(z) = \frac{i}{2}z$ be a biholomorphic map from U' to the unit disc. Then the transformation formula for the Bergman kernels implies that

$$\begin{aligned} K_{U'}(z,\zeta) &= \frac{1}{\pi}\frac{f'(z)\overline{f'(\zeta)}}{\left(1 - f(z)\overline{f(\zeta)}\right)^2} \\ &= \frac{1}{\pi}\frac{\frac{i}{2}\frac{-i}{2}}{\left(1 - \frac{iz}{2}\frac{-i\overline{\zeta}}{2}\right)^2} \\ &= \frac{1}{\pi}\frac{4}{(4 - z\overline{\zeta})^2}. \end{aligned} \tag{1.5}$$

Let $\Omega_\rho = \{z \in \mathbb{C} : \rho < |z| < 1\}$ be a circular annulus. The orthonormal complete set for $H^2(\Omega)$ is given by

$$\begin{aligned} \varphi_{2n-1}(z) &= z^{n-1}\left(\frac{n}{\pi(1-\rho^{2n})}\right)^{\frac{1}{2}}, \qquad n = 1, 2, \dots, \\ \varphi_2(z) &= \frac{1}{z}\left(\frac{1}{-2\pi\ln\rho}\right)^{\frac{1}{2}}, \\ \varphi_{2n}(z) &= \frac{1}{z^n}\left(\frac{1-n}{\pi(1-r^{2(n-1)})}\right)^{\frac{1}{2}}, \qquad n = 2, \dots. \end{aligned}$$

Hence, we have

$$\begin{aligned} K_{\Omega_\rho}(z,\zeta) &= \sum_{n=1}^{\infty}\varphi_n(z)\varphi_n(\overline{\zeta}) \\ &= \frac{1}{\pi z\overline{\zeta}}\left(\mathcal{P}(\ln z\overline{\zeta}) + \frac{\eta_1}{\pi i} - \frac{1}{2\ln\rho}\right), \end{aligned} \tag{1.6}$$

where $\mathcal{P}$ is the Weierstrass function with the periods $\omega_1 = \pi i$, $\omega_2 = \ln \rho$, and $2\eta_1$ is the increment of the Weierstrass ζ-function related to the period ω_1 (see [5]).

On the other hand, the Bergman kernels for domains in $\mathbb{C}^n$ are known in special cases such as the unit ball, the polydisc, the Thullen domain [8], convex domains [6], the Lie ball [10], the minimal ball [17] and so on. For example, the Bergman kernel for the unit ball B in $\mathbb{C}^n$ is

$$K_B(z,\zeta) = \frac{n!}{\pi^n}\,\frac{1}{(1-z\overline{\zeta})^{n+1}}.$$

Suppose that Ω is a bounded domain with C^∞ smooth boundary. The Green function $G_\Omega(z,w)$ and the Bergman kernel $K_\Omega(z,w)$ associated to Ω are related via the following formula, see [1]:

$$K_\Omega(z,\zeta) = -\frac{2}{\pi}\,\frac{\partial^2 G_\Omega(z,\zeta)}{\partial z \partial\overline{\zeta}}. \tag{1.7}$$

2. Bell representations

A holomorphic function $A(z,w)$ on an open set in $\mathbb{C}\times\mathbb{C}$ is called algebraic if there exists a polynomial $P(A(z,w),z,w)=0$.

The kernel $K_\Omega(z,w)$ is algebraic if and only if $K_\Omega(z,w) = R(z,\bar{w})$ where R is a holomorphic algebraic function of $\{(z,\bar{w}) : (z,w)\in\Omega\times\Omega\}$. It is the same as for fixed $b\in\Omega$, $K_\Omega(z,b)$ is an algebraic function of z.

In this section we study the Bell representative domains where the Bergman kernels are algebraic. One can see from (1.3) that it is possible to represent the Bergman kernel for simply connected planar domains via the Riemann map. It is rational if and only if the corresponding Riemann map is rational. For a bounded n-connected domain, the Bergman kernel cannot be rational if $n>1$ (see [2]). Hence, for n-connected domains, it is interesting to study the question, when the Bergman kernel is algebraic even though we cannnot express it explicitly.

Let Ω be an n-connected planar domain and let $f_a\colon \Omega \longrightarrow U$ be the Ahlfors map with $f_a(a)=0$, $f_a'(a)>0$. Then

$$\sum_{k=1}^{n} K_\Omega(z,F_k(\zeta))\overline{F_k'(\zeta)} = f_a'(z)K_U(f_a(z),\zeta)$$

for $z\in\Omega$, $\zeta\in U - f_a(V)$ where $V=\{z\in\Omega :\ f_a'(z)=0\}$ (see [1]).

The following theorem in [3] tells us when the Bergman kernel is algebraic.

Theorem 2.1
Let Ω be an n-connected non-degenerate planar domain. The following statements are equivalent:

1) *The Bergman kernel* $K_\Omega(\cdot,\cdot)$ *is algebraic.*

2) *The Szegő kernel* $S_\Omega(\cdot,\cdot)$ *is algebraic.*

3) *There exists a proper holomorphic map* $f\colon\Omega\longrightarrow U$ *which is algebraic.*

4) *Every proper holomorphic map from* Ω *onto* U *is algebraic.*

Let us consider an example. Let

$$A_r = \left\{z\in\mathbb{C}:\ |z+\tfrac{1}{z}|<r\right\}$$

for $r>2$. Then A_r is a 2-connected domain with real analytic boundary if $r>2$. The algebraic function

$$f_r(z)=\frac{1}{r}\Big(z+\frac{1}{z}\Big)$$

defines a proper holomorphic map from A_r to U which is a 2-sheeted branched covering map and it is algebraic. By the above theorem, the Bergman kernel for A_r is algebraic.

Additionally, the mapping f_r which is a 2-to-1 map from A_r to U extends to a 1-to-1 biholomorphic from every connected component of A_r^c in $\overline{\mathbb{C}}$ onto U^c in $\overline{\mathbb{C}}$. The modulus of A_r is a continuous increasing function of r that approaches to 0 as $r\to 2^+$ and to ∞ as $r\to\infty$. Hence, every 2-connected domain is biholomorphic to one of A_r (see [3]).

This result leads to the conjecture (see [3]) that any n-connected non-degenerate planar domain Ω is biholomorphic to a domain

$$\left\{z\in\mathbb{C}:\ \left|z+\sum_{k=1}^{n-1}\frac{a_k}{z-b_k}\right|<r\right\}$$

with $a_k,b_k\in\mathbb{C}$, $r>0$. Such a domain is called Bell representation and this conjecture is solved in [13]. Let

$$(a,b)=(a_1,a_2,\dots,a_{n-1},b_1,b_2,\dots,b_{n-1})\in\mathbb{C}^{2n-2}$$

and the corresponding domain

$$W_{a,b}=\left\{z\in\mathbb{C}:\ \left|z+\sum_{k=1}^{n-1}\frac{a_k}{z-b_k}\right|<1\right\},\qquad a_k,b_k\in\mathbb{C}.$$

Theorem 2.2 ([13])
Let Ω *be a non-degenerate* n*-connected planar domain with* $n>1$*. Then* Ω *is biholomorphic to a domain* $W_{a,b}$.

The Bergman kernel associated with $W_{a,b}$ is algebraic since $f\colon W_{a,b}\longrightarrow U$ defined by

$$f_{a,b}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is an algebraic proper holomorphic map. To describe domains which possess algebraic proper holomorphic maps onto the unit disc is an important task in the problem of the equivalence between domains. Let

$$B_n = \{(a,b) \in \mathbb{C}^{2n-2} :\ W_{a,b} \text{ is an } n\text{-connected planar domain}\}.$$

B_n is called the coefficient body for n-connected canonical domains. In [14], B_n is explicitly figured out.

Theorem 2.3 ([14])
For $a \in \mathbb{C}$, let $a' \in \mathbb{C}$ be such that $(a')^2 = a$. Then,

$$B_2 = \{(a,b) \in \mathbb{C}^2 :\ a \neq 0,\ |b + 2a'| < 1,\ |b - 2a'| < 1\}.$$

Fix $(a,b) \in B_n$ and let $W_{a,b}$ be the corresponding n-connected canonical domain. Let $E(W_{a,b})$ be the leaf in B_n for $W_{a,b}$ consisting of all the points which correspond to n-connected canonical domains biholomorphically equivalent to $W_{a,b}$.

Theorem 2.4 ([14])
For $r > 2$,

$$E(A_r) = \left\{(a,b) \in B_2 :\ \left|\frac{4a'}{1 - (\overline{b + 2a'})(b - 2a')}\right| = \frac{4r}{4 + r^2}\right\}.$$

In particular, $E(A_r) \cap \{(a,0) \in \mathbb{C}^2\} = \{(a,0) \in \mathbb{C}^2 :\ |a| = r^{-2}\}$.

Now, we give two examples of points in $E(A_r)$ explaining the above theorems.

Example 2.5
For any real θ, let $a = r^{-2}e^{i\theta}$ and $a' = r^{-1}e^{i\frac{\theta}{2}}$ be so that $(a')^2 = a$ and $(a,0) \in E(A_r)$. Let f be defined by $f(z) = a'z$. Take $z \in A_r$ so that $\left|z + \frac{1}{z}\right| < r$. Then $f(z) = w$ satisfies

$$\begin{aligned}\left|w + \frac{a}{w}\right| &= \left|a'z + \frac{a}{a'z}\right| = |a'|\left|z + \frac{1}{z}\right| \\ &< |a'|r = 1.\end{aligned}$$

So, f is a biholomorphic map from A_r onto $W_{a,0}$.

Example 2.6
Let $r = 3$, $a = \frac{9}{169}$, and $a' = \frac{3}{13}$ so that $(a')^2 = a$. Then $(a, 2a') \in B_2$ by Theorem 2.3. Also, since $4a' = \frac{12}{13}$, it belongs to $E(A_3)$ by Theorem 2.4.

For $n > 2$ we have the following theorem suggesting the basic idea for describing B_n.

Theorem 2.7 ([15])
B_n is the set of (a, b) such that equation $f'_{a,b}(z) = 0$ has $2n-2$ solutions $c_1, c_2, \ldots, c_{2n-2}$ counted with multiplicities such that $|f_{a,b}(c_j)| < 1$ for every j. In particular, B_n is an open subset of $\mathbb{C}^{2n-2}$.

Now, we give an example for a point in B_3.

Example 2.8
Let $a_1 = a_2 = \frac{-2+\sqrt{20}}{16^2}$ and $b_1 = -b_2 = \frac{1}{16}$. Then $(a_1, a_2, b_1, b_2) \in B_3$. In fact $\left\{ \pm\frac{\sqrt{3+\sqrt{20}}}{16}, \pm\frac{\sqrt{5-\sqrt{20}}}{16} i \right\}$ is the set of critical points of $f_{a,b}$ and $|f_{a,b}| < 1$ at each critical point.

3. Conformal equivalence between domains

In the previous section, we get the biholomorphic equivalence of any n-connected domain and a Bell representation while we studied the algebraicity property of the Bergman kernel. For 2-connected domains, annuli Ω_ρ and Bell representations A_r are two canonical domains. So, it is interesting to demonstrate the equivalence of these domains. In order to check the conformal equivalence of them, we project them onto the unit disc.

Note that Ω_ρ is biholomorphic to A_r for some $r > 2$ if and only if there is a biholomorphic map $T: U \longrightarrow U$ with $T(\{\pm ic_\rho\}) = \{\pm\frac{2}{r}\}$. The Ahlfors map $f_\rho: \Omega_\rho \longrightarrow U$ with $f_\rho(\sqrt{\rho}) = 0$ and $f'_\rho(\sqrt{\rho}) > 0$ maps $\{|z| = \sqrt{\rho}\}$ onto a line segment with endpoints $\pm ic_\rho$. Hence we get the following theorem.

Theorem 3.1 ([12])
Let $\Omega_\rho = \{z \in \mathbb{C} : \rho < |z| < 1\}$ with $0 < \rho < 1$. Ω_ρ is conformally equivalent to A_r, $(r > 2)$ if and only if $r = \frac{2}{c_\rho}$, where

$$c_\rho = \frac{2\sqrt{\rho}\displaystyle\sum_{k=0}^{\infty}(-1)^{\frac{(k+1)}{2}}\frac{\rho^k}{1+\rho^{2k+1}}}{1 + 2\displaystyle\sum_{k=0}^{\infty}(-1)^{\frac{k+2}{2}}\frac{\rho^{2k+1}}{1+\rho^{2k+1}}}.$$

Also, Crowdy [7] got the relation between r and ρ and constructed a conformal mapping from Ω_ρ onto A_r using Schottky–Klein prime functions associated with Ω_ρ.

Deger [9] showed that when $J(z) = \frac{1}{2}(z + \frac{1}{z})$, $\frac{2}{r}J(z)$ is in fact the Ahlfors map for A_r with $\frac{2}{r}J(i) = 0$ and expressed the Bergman kernel for A_r as

$$K_{A_r}(z,w) = C_1 \frac{2k^2 S(z,\bar{w}) + kC(z,\bar{w})D(z,\bar{w}) + C_2}{z\bar{w}\sqrt{1-k^2J(z)^2}\sqrt{1-k^2J(\bar{w})^2}}$$

where $k = (\frac{2}{r})^2$ and C_1, C_2 are constants that depend only on r and $S(z,w)$, $C(z,w)$, $D(z,w)$ are given.

In fact, $\frac{2}{r}J(z) = f_r(z)$ and so f_r is the Ahlfors map for A_r with $f_r(i) = 0$, $f_r'(i) > 0$. The following expression of the Bergman kernel for any 2-connected domain is given in [4].

THEOREM 3.2
The Bergman kernel $K_\Omega(z,w)$ for any 2-connected planar domain Ω is given by

$$\Phi'(z)K_{A_r}(\Phi(z),\Phi(w))\overline{\Phi'(w)}$$

where the biholomorphic map Φ from Ω onto its representative domain A_r satisfies that $\frac{2}{r}J(\Phi(z)) = \lambda f_a(z)$ where $f_a\colon \Omega \longrightarrow U$ is an Ahlfors map for a point a on the median of Ω, $|\lambda| = 1$.

4. Circular multiply connected planar domain

Let the discs

$$D_k = \{z \in \mathbb{C} : \ |z - a_k| < r_k\}, \qquad k = 1,2,\ldots,n$$

be mutually disjoint and let

$$D = \overline{\mathbb{C}} - \bigcup_{k=1}^{n}(D_k \cup \partial D_k)$$

be the complement of these discs to the extended complex plane. The domain D is called a circular multiply connected domain. Let $f\colon D \longrightarrow \Omega$ be a biholomorphic mapping of D onto a bounded domain Ω with C^∞ smooth boundary. Then

$$K_D(z,\zeta) = f'(z)K_\Omega(f(z),f(\zeta))\overline{f'(\zeta)} \qquad z,\zeta \in D.$$

In addition, the Green functions G_D and G_Ω associated with D and Ω respectively, satisfy the identity

$$G_D(z,\zeta) = G_\Omega(f(z),f(\zeta)), \qquad z,\zeta \in D. \tag{4.1}$$

Hence, the Bergman kernel K_D and the Green function $G_D(z,\zeta)$ associated to D are related via

$$K_D(z,\zeta) = -\frac{2}{\pi}\frac{\partial^2 G_D(z,\zeta)}{\partial z \partial\overline{\zeta}}. \tag{4.2}$$

Let $z^*_{(k)}$ denote the inversions with respect to the circles

$$\partial D_k = \{z:\ |z-a_k| = r_k\}, \qquad k = 1,2,\dots,n,$$

given by

$$z^*_{(k)} := \frac{r_k^2}{\overline{z - a_k}} + a_k\,. \tag{4.3}$$

We denote their compositions by:

$$z^*_{(k_s k_{s-1}\cdots k_1)} := \left(z^*_{(k_{s-1}\cdots k_1)}\right)^*_{(k_s)} \tag{4.4}$$

where two adjacent numbers k_j , k_{j+1} $(j = 1,2,\dots,s-1)$ are not equal. Here s represents the number of inversions and is called the level of the mapping.

These are Möbius transformations γ_j, $(j = 0,1,\dots)$ in z or $\overline{z}$ if s is even or odd, respectively. To be precise, they are defined by

$$\begin{aligned}
\gamma_0(z) &:= z, \\
\gamma_1(\overline{z}) &:= z^*_{(1)}, \qquad \gamma_2(\overline{z}) := z^*_{(2)}, \dots, \qquad \gamma_n(\overline{z}) := z^*_{(n)}, \\
\gamma_{n+1}(z) &:= z^*_{(12)}, \quad \gamma_{n+2}(z) := z^*_{(13)}, \dots, \quad \gamma_{n^2}(z) := z^*_{(n,n-1)}, \\
\gamma_{n^2+1}(\overline{z}) &:= z^*_{(121)}, \qquad \text{and so on.}
\end{aligned}$$

The level s of γ_j is not decreasing. The above functions generate a Schottky group $\mathcal{S}$ (see [16]). Let $\mathcal{S}_m = \{z^*_{(k_s k_{s-1}\dots k_1)}:\ k_s \neq m\} \subset \mathcal{S} - \{\gamma_0\}$.

Mityushev and Rogosin [16] constructed the explicit expression for the complex Green function $M_D(z,\zeta)$ associated to D using the above γ_j . The expression for the real Green function $G_D(z,\zeta)$ and the calculation of $\frac{\partial^2 G_D}{\partial z \partial \overline{\zeta}}$ leads to the following expression for the Bergman kernel $K_D(z,\zeta)$.

Theorem 4.1 ([11])
Let

$$\Psi^{(j)}_m(z) := \begin{cases} \dfrac{\gamma_j'(z)}{\gamma_j(z) - a_m} & \text{if level of } \gamma_j \text{ is even,} \\ -\dfrac{\overline{\gamma_j}'(z)}{\overline{\gamma_j(\overline{z}) - a_m}} & \text{if level of } \gamma_j \text{ is odd.} \end{cases} \tag{4.5}$$

The Bergman kernel $K_D(\cdot,\cdot)$ associated to a circular multiply connected planar domain D is given by

$$K_D(z,\zeta) = -\frac{1}{\pi}\sum_{k=1}^{n}\sum_{m=1}^{n} A_m \overline{\sum_{\gamma_j\in\mathcal{S}_m}^{\infty} \Psi_m^{(j)}(\zeta)} \sum_{\gamma_j\in\mathcal{S}_k} \Psi_k^{(j)}(z) - \frac{1}{\pi}\sum_{\gamma_j\in\mathcal{F}} \frac{\overline{\gamma}_j'(z)}{(\overline{\zeta}-\overline{\gamma}_j(z))^2}. \tag{4.6}$$

where A_m are some real constants and $\mathcal{F}$ is the set of γ_j's of the odd level.

EXAMPLE 4.2
We consider the simply connected domain

$$D = \{z \in \overline{\mathbb{C}} : \ |z| > 2\}.$$

Then we have two-element group of inversions

$$\gamma_0(z) = z, \qquad \gamma_1(\overline{z}) = \frac{2^2}{\overline{z}}.$$

The constant A_1 is equal to zero and

$$K_D(z,\zeta) = -\frac{1}{\pi}\frac{\overline{\gamma}_1'(z)}{(\overline{\zeta}-\overline{\gamma}_1(z))^2} = \frac{1}{\pi}\frac{2^2}{(2^2 - z\overline{\zeta}\,)^2}. \tag{4.7}$$

Similarly, for a general circular simply connected domain

$$D = \{z \in \overline{\mathbb{C}} : \ |z-a_1| > r_1\},$$

the Bergman kernel is given by

$$K_D(z,\zeta) = \frac{1}{\pi}\frac{r_1^2}{(r_1^2 - (z-a_1)\overline{\zeta}\,)^2}$$

and hence it is rational.

We find that the Bergman kernel in (4.6) for D matches with the result in (1.5). Therefore, we conclude that for $n = 1$, D is biholomorphic to U with rational biholomorphic map $f(z) = \frac{r_1}{z-a_1}$ and hence $K_D(z,\zeta)$ is rational.

REMARKS
If $n > 1$, $K_D(z,\zeta)$ is not rational. But, $K_D(z,\zeta)$ is algebraic if there is an algebraic proper holomorphic map from D onto U.

Open questions

1. Find a precise description of B_3 in order to make corresponding Bell representations which are canonical 3-connected domains.
2. Find a relation between the expression (1.6) of the Bergman kernel associated with an annulus and the expression (4.6) of the Bergman kernel associated with a circular doubly connected planar domain.
3. Find relations between circular multiply connected planar domains and Bell representations.

Acknowledgments

The author thanks Prof. Mityushev for helpful discussions.

References

[1] S.R. Bell, *The Cauchy transform, potential theory, and conformal mapping*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[2] S.R. Bell, *Complexity of the classical kernel functions of potential theory*, Indiana Univ. Math. J. **44** (1995), no. 4, 1337-1369.

[3] S.R. Bell, *Finitely generated function fields and complexity in potential theory in the plane*. Duke Math. J. **98** (1999), no. 1, 187-207.

[4] S. Bell, E. Deger, T. Tegtmeyer, *A Riemann mapping theorem for two-connected domains in the plane*, Computational Methods and Function Theory, to appear.

[5] S. Bergman, *The kernel function and conformal mapping*, second, revised edition, Mathematical Surveys, No. **V**, American Mathematical Society, Providence, 1970.

[6] H.P. Boas, S. Fu, E.J. Straube, *The Bergman kernel function: explicit formulas and zeroes*, Proc. Amer. Math. Soc. **127** (1999), no. 3, 805-811.

[7] D. Crowdy, *Conformal mappings from annuli to canonical doubly connected Bell representations*, J. Math. Anal. Appl. **340** (2008), no. 1, 669-674.

[8] J.P. D'Angelo, *An explicit computation of the Bergman kernel function*, J. Geom. Anal. **4** (1994), no. 1, 23-34.

[9] E. Deger, *A biholomorphism from the Bell representative domain onto an annulus and kernel functions*, Purdue Univ. PhD thesis, 2007.

[10] L.K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Translations of Mathematical Monographs **6**, American Mathematical Society, Providence, 1979.

[11] M. Jeong, V. Mityushev, *The Bergman kernel for circular multiply connected domains*, Pacific J. Math. **233** (2007), no. 1, 145-157.

[12] M. Jeong, J. Oh, M. Taniguchi, *Equivalence problem for annuli and Bell representations in the plane*, J. Math. Anal. Appl. **325** (2007), no. 2, 1295-1305.

[13] M. Jeong, M. Taniguchi, *Bell representations of finitely connected planar domains*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2325-2328.

[14] M. Jeong, M. Taniguchi, *Algebraic kernel functions and representation of planar domains*, J. Korean Math. Soc. **40** (2003), no. 3, 447-460.

[15] M. Jeong, M. Taniguchi, *The coefficient body of Bell representations of finitely connected planar domains*, J. Math. Anal. Appl. **295** (2004), no. 2, 620-632.

[16] V. Mityushev, S.V. Rogosin, *Constructive methods for linear and nonlinear boundary value problems for analytic functions. Theory and applications*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **108**, Chapman & Hall/CRC, Boca Raton, 2000.

[17] K. Oeljeklaus, P. Pflug, E.H. Youssfi, *The Bergman kernel of the minimal ball and applications*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 3, 915-928.

Department of Mathematics
The University of Suwon
Suwon P.O.Box 77
Kyungkido 440-600
Korea
E-mail: mjeong@suwon.ac.kr

Received: 15 June 2008; final version: 21 September 2008;
available online: 14 October 2008.

Annales Academiae Paedagogicae Cracoviensis
Folia 63 Studia Mathematica VII

P.A. Krutitskii, N.Ch. Krutitskaya

The harmonic Dirichlet problem in a planar domain with cracks

Abstract. The harmonic Dirichlet problem in a planar domain with smooth cracks of an arbitrary shape is considered in case, when the solution is not continuous at the ends of the cracks. The well-posed formulation of the problem is given, theorems on existence and uniqueness of a classical solution are proved, the integral representation for a solution is obtained. With the help of the integral representation, the properties of the solution are studied. It is proved that a weak solution of the Dirichlet problem in question does not typically exist, though the classical solution exists.

1. Introduction

Boundary value problems in planar domains with cracks are widely used in physics and in mechanics, and not only in mechanics of solids, but in fluid mechanics as well, where cracks (or cuts) model wings or screens in fluids. Integral representation of a classical solution to the harmonic Dirichlet problem in a planar domain with cracks of an arbitrary shape has been obtained by the method of integral equations in [5, 4, 3, 2, 6] in case when the solution is assumed to be continuous at the ends of the cracks. In the present paper this problem is considered in case when the solution is not continuous at the ends of the cracks. The well-posed formulation of the boundary value problem is given, theorems on existence and uniqueness of a classical solution are proved, the integral representation for a classical solution is obtained. Moreover, properties of the solution are studied with the help of this integral representation. It appears that the classical solution to the Dirichlet problem considered in the present paper exists, while the weak solution typically does not exist, though both the cracks and the functions specified in the boundary conditions are smooth enough. This result follows from the fact that the square of the gradient of a classical solution basically is not itegrable near the ends of the cracks, since singularities of the gradient are rather strong there. This

AMS (2000) Subject Classification: 35J05, 35J25, 31A25.
The research has been partly supported by the RFBR grants 08-01-00082, 09-01-00025.

result is very important for numerical analysis, when finite element and finite difference methods are used to obtain numerical solution. To use difference methods for numerical analysis one has to localize all strong singularities first and next to use difference method in a domain excluding the neighbourhoods of the singularities.

2. Formulation of the problem

By an open curve we mean a simple smooth non-closed arc of finite length without self-intersections [8].

In a plane with Cartesian coordinates $x = (x_1, x_2) \in \mathbb{R}^2$ we consider a connected domain $\mathcal{D}$ bounded by simple closed curves $\Gamma_1^2, \ldots, \Gamma_{N_2}^2$ of class $C^{2,\lambda}$, $\lambda \in (0,1]$. It is assumed that the curves $\Gamma_1^2, \ldots, \Gamma_{N_2}^2$ do not have common points. We set $\Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2$, therefore $\partial\mathcal{D} = \Gamma^2$. We will consider both the case of an exterior domain $\mathcal{D}$ and the case of an interior domain $\mathcal{D}$, when the curve Γ_1^2 encloses all others. In the domain $\mathcal{D}$ we consider disjoint open curves $\Gamma_1^1, \ldots, \Gamma_{N_1}^1$ of class $C^{2,\lambda}$. We set $\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1$, so $\Gamma^1 \subset \mathcal{D}$. We assume that points of the curves Γ^1, including endpoints, are interior points of the domain $\mathcal{D}$. In other words, it is assumed that the closed curves Γ^2 and the open curves Γ^1 do not have any common points, moreover, endpoints of Γ^1 do not belong to Γ^2. We set $\Gamma = \Gamma^1 \cup \Gamma^2$.

We assume that each curve Γ_n^j is parametrized by the arc length s:

$$\Gamma_n^j = \{x : \ x = x(s) = (x_1(s), x_2(s)),\ s \in [a_n^j, b_n^j]\}, \qquad n = 1, \ldots, N_j,\ j = 1, 2,$$

so that $a_1^1 < b_1^1 < \ldots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \ldots < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is placed to the right when the parameter s increases on Γ_n^2. The points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except the points a_n^2, b_n^2, which correspond to the same point x for $n = 1, \ldots, N_2$. Further on, the sets of the intervals

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \qquad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \qquad \bigcup_{j=1}^{2} \bigcup_{n=1}^{N_j} [a_n^j, b_n^j]$$

on the Os-axis will be denoted by Γ^1, Γ^2 and Γ too.

For $j = 0, 1$ and $r \in [0, 1]$ set

$$C^{j,r}(\Gamma_n^2) = \{\mathcal{F}(s) : \mathcal{F}(s) \in C^{j,r}[a_n^2, b_n^2],\ \mathcal{F}^{(m)}(a_n^2) = \mathcal{F}^{(m)}(b_n^2),\ m = 0, \ldots, j\}$$

and

$$C^{j,r}(\Gamma^2) = \bigcap_{n=1}^{N_2} C^{j,r}(\Gamma_n^2).$$

The tangent vector to Γ at the point $x(s)$, in the direction of growth of the parameter of s, will be denoted by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, while the normal vector coinciding with τ_x after counterclockwise rotation by the angle of $\frac{\pi}{2}$, will be denoted by $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$. According to the chosen parametrization $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Thus, $\mathbf{n}_x$ is the interior normal to $\mathcal{D}$ on Γ^2. By X we denote the point set consisting of the endpoints of Γ^1:

$$X = \bigcup_{n=1}^{N_1} (x(a_n^1) \cup x(b_n^1)).$$

Let the plane be cut along Γ^1. We consider Γ^1 as a set of cracks (or cuts). The side of the crack Γ^1, which is situated on the left when the parameter s increases, will be denoted by $(\Gamma^1)^+$, while the opposite side will be denoted by $(\Gamma^1)^-$.

We say that the function $u(x)$ belongs to the smoothness class $\mathbf{K}_1$, if

1. $u \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, $\nabla u \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$;
2. in the neighbourhood of any point $x(d) \in X$ the equality

$$\lim_{r \to +0} \int_{\partial S(d,r)} u(x) \frac{\partial u(x)}{\partial \mathbf{n}_x}\, dl = 0 \tag{1}$$

holds, where the curvilinear integral of the first kind is taken over a circle $\partial S(d,r)$ of radius r with the center in the point $x(d)$, $\mathbf{n}_x$ is a normal in the point $x \in \partial S(d,r)$, and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

Remark 1

By $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X)$ we denote the class of functions continuous in $\overline{\mathcal{D}} \setminus \Gamma^1$, which are continuously extendable to the sides of the cracks $\Gamma^1 \setminus X$ from the left and from the right, but their limit values on $\Gamma^1 \setminus X$ can be different from the left and from the right, so that these functions may have a jump on $\Gamma^1 \setminus X$. To obtain the definition of the class $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$ we have to replace $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X)$ by $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$ and $\overline{\mathcal{D}} \setminus \Gamma^1$ by $\mathcal{D} \setminus \Gamma^1$ in the previous sentence.

Problem $\mathbf{D}_1$

Find a function $u(x)$ from $\mathbf{K}_1$, so that $u(x)$ satisfies Laplace equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0, \tag{2a}$$

in $\mathcal{D} \setminus \Gamma^1$ and satisfies the boundary conditions

$$u(x)\big|_{x(s)\in(\Gamma^1)^+} = F^+(s),\ u(x)\big|_{x(s)\in(\Gamma^1)^-} = F^-(s),\ u(x)\big|_{x(s)\in\Gamma^2} = F(s). \tag{2b}$$

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity:

$$|u(x)| \le \text{const}, \qquad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{2c}$$

All conditions of the Problem $\mathbf{D}_1$ must be satisfied in a classical sense. The boundary conditions (2b) on Γ^1 must be satisfied in the interior points of Γ^1, their validity at the ends of Γ^1 is not required.

THEOREM 1
If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, *then there is no more than one solution to the problem* $\mathbf{D}_1$.

It is enough to prove that the homogeneous Problem $\mathbf{D}_1$ admits the trivial solution only. The proof will be given for an interior domain $\mathcal{D}$. Let $u^0(x)$ be a solution to the homogeneous Problem $\mathbf{D}_1$ with $F^+(s) \equiv F^-(s) \equiv 0$, $F(s) \equiv 0$. Let $S(d,\varepsilon)$ be a disc of small enough radius ε, with the center in the point $x(d)$ ($d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$). Let $\Gamma^1_{n,\varepsilon}$ be a set consisting of such points of the curve Γ^1_n which do not belong to discs $S(a^1_n, \varepsilon)$ and $S(b^1_n, \varepsilon)$. We choose a number ε_0 so small that the following conditions are satisfied:

1) for any $0 < \varepsilon \le \varepsilon_0$ the set of points $\Gamma^1_{n,\varepsilon}$ is a unique non-closed arc for each $n = 1, ..., N_1$;
2) the points belonging to $\Gamma \setminus \Gamma^1_n$ are placed outside the discs $S(a^1_n, \varepsilon_0)$, $S(b^1_n, \varepsilon_0)$ for any $n = 1, ..., N_1$;
3) discs of radius ε_0 with centers in different ends of Γ^1 do not intersect.

Set

$$\Gamma^{1,\varepsilon} = \bigcup_{n=1}^{N_1} \Gamma^1_{n,\varepsilon}, \qquad S_\varepsilon = \bigcup_{n=1}^{N_1} [S(a^1_n, \varepsilon) \cup S(b^1_n, \varepsilon)], \qquad \mathcal{D}_\varepsilon = \mathcal{D} \setminus \Gamma^{1,\varepsilon} \setminus S_\varepsilon \, .$$

Since $\Gamma^2 \in C^{2,\lambda}$, $u^0(x) \in C^0(\overline{\mathcal{D}} \setminus \Gamma^1)$ (remind that $u^0(x) \in \mathbf{K}_1$), and since $u^0|_{\Gamma^2} = 0 \in C^{2,\lambda}(\Gamma^2)$, and due to the theorem on regularity of solutions of elliptic equations near the boundary [1], we obtain: $u^0(x) \in C^1(\overline{\mathcal{D}} \setminus \Gamma^1)$. Since $u^0(x) \in \mathbf{K}_1$, we observe that $u^0(x) \in C^1(\overline{\mathcal{D}}_\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_0]$. By $C^1(\overline{\mathcal{D}}_\varepsilon)$ we mean $C^1(\mathcal{D}_\varepsilon \cup \Gamma^2 \cup (\Gamma^{1,\varepsilon})^+ \cup (\Gamma^{1,\varepsilon})^- \cup \partial S_\varepsilon)$. Since the boundary of the domain $\mathcal{D}_\varepsilon$ is piecewise smooth, we write down Green's formula [10, p. 328] for the function $u^0(x)$:

$$\|\nabla u^0\|^2_{L_2(\mathcal{D}_\varepsilon)} = \int_{\Gamma^{1,\varepsilon}} (u^0)^+ \left(\frac{\partial u^0}{\partial \mathbf{n}_x}\right)^+ ds - \int_{\Gamma^{1,\varepsilon}} (u^0)^- \left(\frac{\partial u^0}{\partial \mathbf{n}_x}\right)^- ds$$
$$- \int_{\Gamma^2} u^0 \frac{\partial u^0}{\partial \mathbf{n}_x} ds + \int_{\partial S_\varepsilon} u^0 \frac{\partial u^0}{\partial \mathbf{n}_x} dl.$$

The exterior (with respect to $\mathcal{D}_\varepsilon$) normal on ∂S_ε at the point $x \in \partial S_\varepsilon$ is denoted by $\mathbf{n}_x$. By the superscripts $+$ and $-$ we denote the limit values of functions on $(\Gamma^1)^+$ and on $(\Gamma^1)^-$, respectively. Since $u^0(x)$ satisfies the homogeneous boundary condition (2b) on Γ, we observe that $u^0|_{\Gamma^2} = 0$ and $(u^0)^{\pm}|_{\Gamma^{1,\varepsilon}} = 0$ for any $\varepsilon \in (0, \varepsilon_0]$. Therefore

$$\|\nabla u^0\|^2_{L_2(\mathcal{D}_\varepsilon)} = \int\limits_{\partial S_\varepsilon} u^0 \frac{\partial u^0}{\partial \mathbf{n}_x}\, dl, \qquad \varepsilon \in (0, \varepsilon_0].$$

Setting $\varepsilon \to +0$, taking into account that $u^0(x) \in \mathbf{K}_1$ and using the relationship (1), we obtain:

$$\|\nabla u^0\|^2_{L_2(\mathcal{D}\setminus\Gamma^1)} = \lim_{\varepsilon\to+0} \|\nabla u^0\|^2_{L_2(\mathcal{D}_\varepsilon)} = 0.$$

From the homogeneous boundary conditions (2b) we conclude that $u^0(x) \equiv 0$ in $\mathcal{D} \setminus \Gamma^1$, where $\mathcal{D}$ is an interior domain. If $\mathcal{D}$ is an exterior domain, then the proof is analogous, but we have to use the condition (2c) and the theorem on behaviour of the gradient of a harmonic function at infinity [10, p. 373]. The maximum principle cannot be used for the proof of the theorem even in the case of the interior domain $\mathcal{D}$, since the solution to the problem may not satisfy the boundary condition (2b) at the ends of the cracks, and it may not be continuous at the ends of the cracks.

3. Existence of a classical solution

Let us turn to solving the Problem $\mathbf{D}_1$. Consider the double layer harmonic potential with the density $\mu(s)$ specified at the open arcs Γ^1:

$$w[\mu](x) = -\frac{1}{2\pi} \int\limits_{\Gamma^1} \mu(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x - y(\sigma)|\, d\sigma. \tag{3}$$

THEOREM 2
Let $\Gamma^1 \in C^{1,\lambda}$, $\lambda \in (0,1]$. Let $S(d,\varepsilon)$ be a disc of a small enough radius ε with the center in the point $x(d)$ ($d = a^1_n$ or $d = b^1_n$, $n = 1, ..., N_1$).

I. *If $\mu(s) \in C^{0,\lambda}(\Gamma^1)$, then $w[\mu](x) \in C^0(\overline{\mathbb{R}^2 \setminus \Gamma^1} \setminus X)$ and for any $x \in S(d,\varepsilon)$, such that $x \notin \Gamma^1$, the inequality holds: $|w[\mu](x)| \le \mathrm{const}$.*

II. *If $\mu(s) \in C^{1,\lambda}(\Gamma^1)$, then*

1) $\nabla w[\mu](x) \in C^0(\overline{\mathbb{R}^2 \setminus \Gamma^1} \setminus X)$;
2) *for any $x \in S(d,\varepsilon)$, such that $x \notin \Gamma^1$, the formulae hold*

$$\frac{\partial w[\mu](x)}{\partial x_1} = \frac{1}{2\pi} \frac{\mp\mu(d)}{|x - x(d)|} \sin\psi(x, x(d)) + \Omega_1(x),$$

$$\sin\psi(x, x(d)) = \frac{x_2 - x_2(d)}{|x - x(d)|},$$

$$\frac{\partial w[\mu](x)}{\partial x_2} = \frac{1}{2\pi}\frac{\pm\mu(d)}{|x-x(d)|}\cos\psi(x,x(d)) + \Omega_2(x),$$

$$\cos\psi(x,x(d)) = \frac{x_1 - x_1(d)}{|x-x(d)|},$$

$$|\Omega_j(x)| \le \mathrm{const}\cdot\ln\frac{1}{|x-x(d)|}, \qquad j=1,2,$$

the upper sign in the formulae is taken if $d = a_n^1$*, while the lower sign is taken if* $d = b_n^1$*;*

3) *for* $w[\mu](x)$ *the relationship holds*

$$\lim_{\varepsilon\to+0}\int\limits_{\partial S(d,\varepsilon)} w[\mu](x)\frac{\partial w[\mu](x)}{\partial \mathbf{n}_x}\,dl = 0,$$

where the curvilinear integral of the first kind is taken over the circle $\partial S(d,\varepsilon)$*; in addition,* $\mathbf{n}_x = (-\cos\psi(x,x(d)), -\sin\psi(x,x(d)))$ *is the normal at* $x\in\partial S(d,\varepsilon)$*, directed to the center of the circle;*

4) $|\nabla w[\mu](x)|$ *belongs to* $L_2(S(d,\varepsilon))$ *for any small* $\varepsilon>0$ *if and only if* $\mu(d)=0$.

Class $C^0(\overline{\mathbb{R}^2\setminus\Gamma^1}\setminus X)$ is defined in the remark to the definition of the class $\mathbf{K}_1$ (Remark 1), if we set $\mathcal{D}=\mathbb{R}^2$. The proof of the theorem is based on the representation of a double layer potential in the form of the real part of the Cauchy integral with the real density $\mu(\sigma)$:

$$w[\mu](x) = -\operatorname{Re}\Phi(z), \qquad \Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma^1}\mu(\sigma)\frac{dt}{t-z}, \qquad z = x_1 + ix_2\,,$$

where $t = t(\sigma) = (y_1(\sigma)+iy_2(\sigma))\in\Gamma^1$. If $\mu(\sigma)\in C^{1,\lambda}(\Gamma^1)$, then for $z\notin\Gamma^1$:

$$\frac{d\Phi(z)}{dz} = -w'_{x_1} + iw'_{x_2}$$
$$= -\frac{1}{2\pi i}\left(\sum_{n=1}^{N_1}\left\{\frac{\mu(b_n^1)}{t(b_n^1)-z} - \frac{\mu(a_n^1)}{t(a_n^1)-z}\right\} - \int\limits_{\Gamma^1}\frac{e^{-i\alpha(\sigma)}\mu'(\sigma)}{t-z}\,dt\right).$$

Points I, II.1) and II.2) of Theorem 2 follow from these formulae and from the properties of Cauchy integrals, presented in [8]. Points II.3) and II.4) can be proved by direct verification using points I, II.1) and II.2).

We will construct a solution to the Problem $\mathbf{D}_1$ in assumption that $F^+(s), F^-(s)\in C^{1,\lambda}(\Gamma^1)$, $\lambda\in(0,1]$, $F(s)\in C^0(\Gamma^2)$. We will look for a solution to the Problem $\mathbf{D}_1$ of the form

$$u(x) = -w[F^+ - F^-](x) + v(x), \tag{4}$$

where $w[F^+ - F^-](x)$ is the double layer potential (3), in which

$$\mu(\sigma) = F^+(\sigma) - F^-(\sigma).$$

The potential $w[F^+ - F^-](x)$ satisfies the Laplace equation (2a) in $\mathcal{D} \setminus \Gamma^1$ and belongs to the class $\mathbf{K}_1$ according to Theorem 2. Limit values of the potential $w[F^+ - F^-](x)$ on $(\Gamma^1)^{\pm}$ are given by the formula

$$w[F^+ - F^-](x)\big|_{x(s)\in(\Gamma^1)^{\pm}} = \mp \frac{F^+(s) - F^-(s)}{2} + w[F^+ - F^-](x(s)),$$

where $w[F^+ - F^-](x(s))$ is the direct value of the potential on Γ^1.

The function $v(x)$ in (4) must be a solution to the following problem.

Problem **D**

Find a function $v(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, which satisfies the Laplace equation (2a) in the domain $\mathcal{D} \setminus \Gamma^1$ and satisfies the boundary conditions

$$v(x)\big|_{x(s)\in\Gamma^1} = \frac{F^+(s) + F^-(s)}{2} + w[F^+ - F^-](x(s)) = f(s),$$
$$v(x)\big|_{x(s)\in\Gamma^2} = F(s) + w[F^+ - F^-](x(s)) = f(s).$$

If $x(s) \in \Gamma^1$, then $w[F^+ - F^-](x(s))$ is the direct value of the potential on Γ^1.

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity:

$$|v(x)| \leq \text{const}, \qquad |x| = \sqrt{x_1^2 + x_2^2} \to \infty.$$

All conditions of the Problem **D** have to be satisfied in the classical sense. Obviously, $w[F^+ - F^-](x(s)) \in C^0(\Gamma^2)$. It follows from [7, Lemma 4(1)] that $w[F^+ - F^-](x(s)) \in C^{1,\frac{\lambda}{4}}(\Gamma^1)$ (here by $w[F^+ - F^-](x(s))$ we mean the direct value of the potential on Γ^1). So, $f(s) \in C^{1,\frac{\lambda}{4}}(\Gamma^1)$ and $f(s) \in C^0(\Gamma^2)$.

We will look for the function $v(x)$ in the smoothness class $\mathbf{K}$. We say that the function $v(x)$ belongs to the smoothness class $\mathbf{K}$ if

1. $v(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, $\nabla v \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$, where X is the set consisting of the endpoints of Γ^1;

2. in a neghbourhood of any point $x(d) \in X$ the inequality

$$|\nabla v| \leq \mathcal{C}|x - x(d)|^{\delta}$$

holds for some constants $\mathcal{C} > 0$, $\delta > -1$, where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

The definition of the functional class $C^0(\overline{\mathcal{D}\setminus\Gamma^1}\setminus\Gamma^2\setminus X)$ is given in the remark to the definition of the smoothness class $\mathbf{K}_1$ (Remark 1). Clearly, $\mathbf{K}\subset\mathbf{K}_1$.

It can be verified directly that if $v(x)$ is a solution to the Problem $\mathbf{D}$ in the class $\mathbf{K}$, then the function (4) is a solution to the Problem $\mathbf{D}_1$.

Theorem 3
Let $\Gamma\in C^{2,\frac{\lambda}{4}}$, $f(s)\in C^{1,\frac{\lambda}{4}}(\Gamma^1)$, $\lambda\in(0,1]$, $f(s)\in C^0(\Gamma^2)$. Then the solution to the Problem $\mathbf{D}$ in the smoothness class $\mathbf{K}$ exists and is unique.

Theorem 3 has been proved in the following papers: 1) in [5, 4], if $\mathcal{D}$ is an interior domain; 2) in [3], if $\mathcal{D}$ is an exterior domain and $\Gamma^2\neq\emptyset$; 3) in [2, 6], if $\Gamma^2=\emptyset$ and so $\mathcal{D}=\mathbb{R}^2$ is an exterior domain. In all these papers, the integral representations for the solution to the Problem $\mathbf{D}$ in the class $\mathbf{K}$ are obtained in the form of potentials, densities of which are defined by the uniquely solvable Fredholm integro-algebraic equations of the second kind and index zero. Uniqueness of a solution to the Problem $\mathbf{D}$ is proved either by the maximum principle or by the method of energy (integral) identities. In the latter case we take into account that a solution to the problem belongs to the class $\mathbf{K}$. Note that the Problem $\mathbf{D}$ is a particular case of more general boundary value problems studied in [4, 3, 2, 6].

Note that Theorem 3 holds if $\Gamma\in C^{2,\lambda}$, $F^+(s),F^-(s)\in C^{1,\lambda}(\Gamma^1)$, $\lambda\in(0,1]$, $F(s)\in C^0(\Gamma^2)$. From Theorems 2, 3 we obtain the solvability of the problem $\mathbf{D}_1$.

Theorem 4
Let $\Gamma\in C^{2,\lambda}$, $F^+(s),F^-(s)\in C^{1,\lambda}(\Gamma^1)$, $\lambda\in(0,1]$, $F(s)\in C^0(\Gamma^2)$. Then a solution to the Problem $\mathbf{D}_1$ exists and is given by the formula (4), where $v(x)$ is a unique solution to the Problem $\mathbf{D}$ in the class $\mathbf{K}$, ensured by Theorem 3.

Remark 2
Let us check that the solution to the Problem $\mathbf{D}_1$ given by formula (4) satisfies condition (1). Let $d=a_n^1$ or $d=b_n^1$ $(n=1,...,N_1)$ and r be small enough. Then substituting (4) in the integral in (1) we obtain

$$\int\limits_{\partial S(d,r)} u(x)\frac{\partial u(x)}{\partial\mathbf{n}_x}\,dl=\int\limits_{\partial S(d,r)} w(x)\frac{\partial w(x)}{\partial\mathbf{n}_x}\,dl-\int\limits_{\partial S(d,r)} w(x)\frac{\partial v(x)}{\partial\mathbf{n}_x}\,dl$$
$$-\int\limits_{\partial S(d,r)} v(x)\frac{\partial w(x)}{\partial\mathbf{n}_x}\,dl+\int\limits_{\partial S(d,r)} v(x)\frac{\partial v(x)}{\partial\mathbf{n}_x}\,dl.$$

If $r\to0$, then the first term tends to zero by Theorem 2(II.3). As mentioned above, $v(x)\in\mathbf{K}\subset\mathbf{K}_1$, therefore the condition (1) holds for the function $v(x)$, so the fourth term tends to zero as $r \to 0$. The second term tends to zero as $r \to 0$, since $w(x)$ is bounded at the ends of Γ^1 according to Theorem 2(I), and since $v(x)$ satisfies condition 2) in the definition of the class $\mathbf{K}$. Noting that $v(x)$ is continuous at the ends of Γ^1 due to the definition of the class $\mathbf{K}$, and using Theorem 2(II.2) for calculation of $\frac{\partial w(x)}{\partial \mathbf{n}_x}$ in the third term, we deduce that the third term tends to zero when $r \to 0$ as well. Consequently, the equality (1) holds for the solution to the Problem $\mathbf{D}_1$ constructed in Theorem 4.

Uniqueness of a solution to the Problem $\mathbf{D}_1$ follows from Theorem 1. The solution to the Problem $\mathbf{D}_1$ found in Theorem 4 is, in fact, a classical solution. Let us discuss, under which conditions this solution to the Problem $\mathbf{D}_1$ is not a weak solution.

4. Non-existence of a weak solution

Let $u(x)$ be a solution to the Problem $\mathbf{D}_1$ defined in Theorem 4 by the formula (4). Consider the disc $S(d,\varepsilon)$ with the center in the point $x(d) \in X$ and of radius $\varepsilon > 0$ ($d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$). In doing so, ε is a fixed positive number, which can be taken small enough. Since $v(x) \in \mathbf{K}$, we have $v(x) \in L_2(S(d,\varepsilon))$ and $|\nabla v(x)| \in L_2(S(d,\varepsilon))$ (this follows from the definition of the smoothness class $\mathbf{K}$). Let $x \in S(d,\varepsilon)$ and $x \notin \Gamma^1$. It follows from (4) that $|\nabla w[\mu](x)| \le |\nabla u(x)| + |\nabla v(x)|$, whence

$$\begin{aligned} |\nabla w[\mu](x)|^2 &\le |\nabla u(x)|^2 + |\nabla v(x)|^2 + 2|\nabla u(x)| \cdot |\nabla v(x)| \\ &\le 2(|\nabla u(x)|^2 + |\nabla v(x)|^2). \end{aligned}$$

Assume that $|\nabla u(x)|$ belongs to $L_2(S(d,\varepsilon))$; then, integrating this inequality over $S(d,\varepsilon)$, we obtain

$$\|\nabla w\|^2\big|_{L_2(S(d,\varepsilon))} \le 2\big(\|\nabla u\|^2\big|_{L_2(S(d,\varepsilon))} + \|\nabla v\|^2\big|_{L_2(S(d,\varepsilon))}\big).$$

Consequently, if $|\nabla u(x)| \in L_2(S(d,\varepsilon))$, then $|\nabla w| \in L_2(S(d,\varepsilon))$. However, according to Theorem 2, if $F^+(d) - F^-(d) \ne 0$, then $|\nabla w|$ does not belong to $L_2(S(d,\varepsilon))$. Therefore, if $F^+(d) \ne F^-(d)$, then our assumption that $|\nabla u| \in L_2(S(d,\varepsilon))$ does not hold, i.e., $|\nabla u| \notin L_2(S(d,\varepsilon))$. Thus, if among numbers $a_1^1, ..., a_{N_1}^1$, $b_1^1, ..., b_{N_1}^1$ there exists such a number d that $F^+(d) \ne F^-(d)$, then for some $\varepsilon > 0$ we have $|\nabla u| \notin L_2(S(d,\varepsilon)) = L_2(S(d,\varepsilon) \setminus \Gamma^1)$, so $u \notin W_2^1(S(d,\varepsilon) \setminus \Gamma^1)$, where W_2^1 is a Sobolev space of functions from L_2, which have generalized derivatives from L_2. We have proved the following result.

THEOREM 5
Let conditions of Theorem 4 be satisfied and assume that there exists a number $d \in \{a_1^1, .., a_{N_1}^1, b_1^1, ..., b_{N_1}^1\}$ *such that* $F^+(d) \ne F^-(d)$. *Then the solution to the*

Problem $\mathbf{D}_1$ *, ensured by Theorem 4, does not belong to* $W_2^1(S(d,\varepsilon)\setminus\Gamma^1)$ *for some* $\varepsilon > 0$*, whence it follows that it does not belong to* $W^1_{2,loc}(\mathcal{D}\setminus\Gamma^1)$*. Here* $S(d,\varepsilon)$ *is a disc of a radius* ε *with the center in the point* $x(d)\in X$*.*

By $W^1_{2,loc}(\mathcal{D}\setminus\Gamma^1)$ we denote the class of functions which belong to W_2^1 on any bounded subdomain of $\mathcal{D}\setminus\Gamma^1$. If conditions of Theorem 5 hold, then the unique solution to the Problem $\mathbf{D}_1$, constructed in Theorem 4, does not belong to $W^1_{2,loc}(\mathcal{D}\setminus\Gamma^1)$, and so it is not a weak solution. We arrive to

Corollary

Let conditions of Theorem 5 be satisfied; then a weak solution to the Problem $\mathbf{D}_1$ *in the class of functions* $W^1_{2,loc}(\mathcal{D}\setminus\Gamma^1)$ *does not exist.*

Remark 3

Even if the number d, mentioned in Theorem 5, does not exist, then the solution $u(x)$ to the Problem $\mathbf{D}_1$, ensured by Theorem 4, may not be a weak solution to the Problem $\mathbf{D}_1$. The Hadamard example of a non-existence of a weak solution to the harmonic Dirichlet problem in a disc with continuous boundary data is given in [9, § 12.5] (the classical solution exists in this example).

Clearly, $L_2(\mathcal{D}\setminus\Gamma^1) = L_2(\mathcal{D})$, since Γ^1 is a set of zero measure.

References

[1] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften Vol. **224**, Springer-Verlag, Berlin – New York, 1977.

[2] P.A. Krutitskii, *A mixed problem for the Laplace equation outside cuts on the plane*, Differential Equations **33** (1997), no. 9, 1184-1193.

[3] P.A. Krutitskii, *The 2-dimensional Dirichlet problem in an external domain with cuts*, Z. Anal. Anwendungen **17** (1998), no. 2, 361-378.

[4] P.A. Krutitskii, *The Dirichlet problem for the two-dimensional Laplace equation in a multiply connected domain with cuts*, Proc. Edinburgh Math. Soc. **43** (2000), no. 2, 325-341.

[5] P.A. Krutitskii, *The integral representation for a solution of the 2-D Dirichlet problem with boundary data on closed and open curves*, Mathematika (London) **47** (2000), no. 1-2, 339-354.

[6] P.A. Krutitskii, *The mixed harmonic problem in an exterior cracked domain with Dirichlet condition on cracks*, Comput. Math. Appl. **50** (2005), no. 5-6, 769-782.

[7] P.A. Krutitskii, V.V. Kolybasova, *On the mixed problem for the dissipative Helmholtz equation in a 2D cracked domain with the Neumann condition on cracks*, Complex Var. Elliptic Equ. **51** (2006), no. 8-11, 1065-1089.

[8] N.I. Muskhelishvili, *Singular integral equations. Boundary problems of functions theory and their applications to mathematical physics*, revised translation from the Russian, edited by J.R.M. Radok, Wolters-Noordhoff Publishing, Groningen, 1972.

[9] S.L. Sobolev, *Некоторые применения функционального анализа в математической физике* (Russian) [Some applications of functional analysis in mathematical physics], third edition, "Nauka", Moscow, 1988.

[10] V.S. Vladimirov, *Уравнения математической физики*, (Russian) [The equations of mathematical physics] fourth edition, "Nauka", Moscow, 1981.

KIAM
Miusskaya Sq. 4
Moscow 117234
Russia

Dept. of Mathematics
Faculty of Physics
Moscow State University
Moscow 117234
Russia
E-mail: conf@math.phys.msu.su

Received: 25 June 2008; final version: 29 September 2008;
available online: 20 November 2008.

Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

Report of Meeting

12th International Conference on Functional Equations and Inequalities, Będlewo, September 7-14, 2008

Contents

The *Twelfth International Conference on Functional Equations and Inequalities* was held from September 7 to 14, 2008 in Będlewo, Poland. The series of ICFEI meetings has been organized by the *Institute of Mathematics of the Pedagogical University of Cracow* since 1984. For the third time, the conference was organized jointly with the *Stefan Banach International Mathematical Center* and hosted by the *Mathematical Research and Conference Center* in Będlewo. As usual, the conference was devoted mainly to various aspects of functional equations and inequalities. A special emphasis was given to applications of functional equations. A Workshop on the latter theme, chaired by Prof. Vladimir Mityushev, followed the regular ICFEI meeting.

The Scientific Committee consisted of Professors Nicole Brillouët-Belluot, Dobiesław Brydak (Honorary Chairman), Janusz Brzdęk (Chairman), Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Marek Cezary Zdun and Jacek Chmieliński (Secretary). The Organizing Committee consisted of Janusz Brzdęk (Chairman), Vladimir Mityushev, Paweł Solarz, Janina Wiercioch and Władysław Wilk.

The 57 participants came from 11 countries: China, Germany, Greece, Hungary, Israel, Japan, Romania, Slovenia, USA, Russia and from Poland.

The Conference was opened on Monday morning, September 8 by Professor Janusz Brzdęk – Chairman of the Scientific and Organizing Committees. This ceremony was followed by the first scientific session chaired by Professor Bogdan Choczewski and the first lecture was given by Professor Roman Ger. Altogether, during 20 scientific sessions 4 lectures and 47 short talks were delivered. They focused on functional equations in a single variable and in several variables, functional inequalities, stability theory, convexity, multifunctions, theory of iteration, means, differential and difference equations, functional equations in functional analysis, functional equations in physics and other topics. Several contributions have been made during special *Problems and Remarks* sessions.

On Tuesday, September 9, a picnic was organized in the park surrounding the Center. On the next day afternoon participants visited Poznań with its old city, baroque parish church and National Museum. In the evening the piano recital was performed by Marek Czerni and Hans-Heinrich Kairies. On Thursday, September 21, a banquet was held in the Palace in Będlewo. It was an occasion to honour Prof. Zoltán Daróczy on the occasion of his 70th birthday, celebrated this year. On the following day a *Flamenco evening* was hosted by Małgorzata Drzał (dance & vocal) and Grzegorz Guzik (guitar).

The ICFEI conference was closed on Saturday, September 13 by Professor Bogdan Choczewski. In the closing address, he gave some concluding information about the meeting and conveyed best regards for the participants from the Honorary Chairman of the ICFEI, Professor Dobiesław Brydak. It was announced that Professor Zsolt Páles joined the ICFEI Scientific Committee and that the 13th ICFEI will be organized in 2009.

On Saturday afternoon, September 13 and Sunday morning, September 14 the Workshop devoted to applications of functional equations was held. 4 sessions were organized with 3 lectures, 2 talks and discussion.

The following part of the report contains abstracts of talks (in alphabetical order of the authors' names), problems and remarks (in chronological order of presentation) and the list of participants (with addresses). It has been compiled by Jacek Chmieliński.

Abstracts of Talks

Marcin Adam *On the double quadratic difference property*

Let X be a real normed space and Y a real Banach space. Denote by $C^n(X,Y)$ the class of n-times continuously differentiable functions $f\colon X \to Y$. We prove that the class C^n has the double quadratic difference property, that is if

$$Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y) \in C^n(X \times X, Y),$$

then there exists exactly one quadratic function $K\colon X \to Y$ such that $f - K \in C^n(X,Y)$.

Mirosław Adamek *On two variable functional inequality and related functional equation*

We present the result stating that the lower semicontinuous solutions of a large class of functional inequalities can be obtained from particular solutions of the related functional equations. Our main theorem reads as follows.

THEOREM
Let $\lambda\colon I^2 \longrightarrow (0,1)$ *be a function and* $n, m\colon I^2 \longrightarrow I$ *be continuous strict means. If there exists a non-constant and continuous solution* $\phi\colon I \longrightarrow \mathbb{R}$ *of the equation*

$$T^{\lambda}_{(n(x,y),m(x,y))}\phi = T^{\lambda}_{(x,y)}\phi, \qquad x, y \in I,$$

then ϕ *is one-to-one, and a lower semicontinuous function* $f\colon I \longrightarrow \mathbb{R}$ *satisfies the inequality*

$$T^{\lambda}_{(n(x,y),m(x,y))}f \leq T^{\lambda}_{(x,y)}f, \qquad x, y \in I,$$

if and only if $f \circ \phi^{-1}$ *is convex on* $\phi(I)$.

This result improves results presented in [1] and [2].

[1] J. Matkowski, M. Wróbel, *A generalized a-Wright convexity and related function equation*, Ann. Math. Silesianae **10** (1996), 7-12.

[2] Zs. Páles, *On two variable functional inequality*, C. R. Math. Rep. Acad. Sci. Canada, **10** (1988), 25-28.

Anna Bahyrycz *A system of functional equations related to plurality functions*

We consider the system of functional equations related to plurality functions:

$$f(x)\cdot f(y) \neq 0_m \implies f(x+y) = f(x)\cdot f(y),$$
$$f(rx) = f(x),$$

where $f\colon \mathbb{R}(n) := [0,+\infty)^n \setminus \{0_n\} \longrightarrow \mathbb{R}(m)$, $n, m \in \mathbb{N}$, $r \in \mathbb{R}(1)$ and

$$x+y := (x_1+y_1, ..., x_k+y_k), \quad x\cdot y := (x_1\cdot y_1, ..., x_k\cdot y_k), \quad rx := (rx_1, ..., rx_k)$$

for $x = (x_1, ..., x_k) \in \mathbb{R}(k)$, $y = (y_1, ..., y_k) \in \mathbb{R}(k)$.

We investigate systems of cones over $\mathbb{R}$, which are the parameter determining the solutions of this system.

Karol Baron *Random-valued functions and iterative equations*

As emphasized in [1; 0.3], iteration is the fundamental technique for solving functional equations of the form

$$F(x, \varphi(x), \varphi \circ f(x, \cdot)) = 0,$$

and iterates usually appear in the formulae for solutions. Moreover, many results may be interpreted in both ways: either as theorems about the behaviour of iterates, or as theorems about solutions of functional equations. In this survey we are interested in formulae of the form

$$\varphi(x) = \text{probability that the sequence } (f^n(x,\cdot))_{n\in\mathbb{N}} \text{ converges and its limit belongs to } B,$$

where the iterates f^n, $n \in \mathbb{N}$, are defined as in [1; 1.4] and B is a Borel set. Such formulae defining solutions of

$$\varphi(x) = \int_\Omega \varphi(f(x,\omega))\mathrm{Prob}(d\omega)$$

are rather new in the theory of iterative functional equations, but as in more classical cases also results involving them may be read in two ways above described.

[1] Marek Kuczma, Bogdan Choczewski, Roman Ger, *Iterative Functional Equations*, Encyclopedia of Mathematics and its Applications, Vol. **32**, Cambridge University Press, Cambridge, 1990.

Nicole Brillouët-Belluot *Some aspects of functional equations in physics* (presented by **Joachim Domsta**)

Functional equations represent a way of modelling problems in physics. The physical problem is often directly stated in terms of one or several functional equations. However, a problem in physics may also be firstly described by a partial differential equation from which we derive a functional equation whose solutions solve the problem.

In this talk, I will present several examples of functional equations modelling physical problems in various fields of physics. In each example, I will mainly explain how the functional equation appears in the physical problem.

Janusz Brzdęk *Fixed point results and stability of functional equations in single variable*

Joint work with Roman Badora.

We show that stability of numerous functional equations in single variable is an immediate consequence of very simple fixed point results. We consider a generalization of the classical Hyers–Ulam stability (as suggested by T. Aoki, D.G. Bourgin and Th.M. Rassias), a modification of it, quotient stability (in the sense of R. Ger), and iterative stability.

Liviu Cădariu *Fixed points method for the generalized stability of monomial functional equations*

Joint work with Viorel Radu.

D.H. Hyers in 1941 gave an affirmative answer to a question of S.M. Ulam, concerning the stability of group homomorphisms in Banach spaces: *Let* E_1 *and* E_2 *be Banach spaces and* $f\colon E_1 \longrightarrow E_2$ *be such a mapping that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{1}$$

for all $x, y \in E_1$ *and a* $\delta > 0$, *that is* f *is* $\delta-$additive. *Then there exists a unique* additive $T\colon E_1 \longrightarrow E_2$, *given by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \qquad x \in E_1\,, \tag{2}$$

which satisfies $\|f(x) - T(x)\| \leq \delta$, $x \in E_1$.

T. Aoki, D. Bourgin and Th.M. Rassias studied the stability problem with unbounded Cauchy differences. Generally, the constant δ in (1) is replaced by a control function, $||\mathcal{D}_f(x,y)|| \leq \delta(x,y)$, where, for example, $\mathcal{D}_f(x,y) = f(x+y) - f(x) - f(y)$ for Cauchy equation. The *stability estimations* are of the form $||f(x) - S(x)|| \leq \varepsilon(x)$, where S *verifies the functional equation* $\mathcal{D}_S(x,y) = 0$, and for $\varepsilon(x)$ explicit formulae are given, which depend on the control δ as well as on the equation.

We use a fixed point method, initiated in [3] and developed, e.g., in [1], to give a generalized Ulam–Hyers stability result for functional equations in single variable and functions defined on groups, with values in sequentially complete locally convex spaces. This result is then used to obtain the generalized stability for some abstract monomial functional equations.

[1] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure and Appl. Math. **4(1)** (2003), Art. 4 (http://jipam.vu.edu.au).

[2] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications vol. **34**, Birkhäuser, Boston – Basel – Berlin, 1998.

[3] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, Cluj-Napoca **IV(1)** (2003), 91-96.

Bogdan Choczewski *Special solutions of an iterative functional inequality of second order*

This a report on a joint work by Dobiesław Brydak, Marek Czerni and the speaker [1].

The inequality reads:

$$\psi[f^2(x)] \leq (p(x) + q(f(x))\psi[f(x)] - p(x)q(x)\psi(x), \tag{1}$$

where ψ is the unknown function. We aim at investigating these continuous solutions of (1) that behave at the fixed point of f like a prescribed "test" function T, in particular, like one from among the functions p, q or f.

Inequality (1) has been first studied by Maria Stopa [2].

[1] D. Brydak, B. Choczewski, M. Czerni, *Asymptotic properties of solutions of some iterative functional inequalities*, Opuscula Math., Volume dedicated to the memory of Professor Andrzej Lasota, in print.

[2] M. Stopa, *On the form of solutions of some iterative functional inequality*, Publ. Math. Debrecen **45** (1994), 371-377.

Jacek Chudziak *On some property of the Gołąb–Schinzel equation*

Let X be a linear space over a field K of real or complex numbers. Given nonempty subset A of X, we say that $a \in A$ is an algebraically interior point to A provided, for every $x \in X \setminus \{0\}$, there is an $r_x > 0$ such that

$$\{a + bx : \ |b| < r_x\} \subset A.$$

By $\mathrm{int}_a A$ we denote the set of all algebraically interior points to A.

We show that, rather surprisingly, in a class of functions $f\colon X \longrightarrow K$ such that $F_f := \{x \in \ X : \ f(x) = 0\} \neq \emptyset$ and $\mathrm{int}_a(X \setminus F_f) \neq \emptyset$, the following two conditions are equivalent:

(i) $f(x + f(x)y) = 0$ if and only if $f(x)f(y) = 0$ for $x, y \in X$;

(ii) $f(x + f(x)y) = f(x)f(y)$ for $x, y \in X$.

Some consequences of this fact are also presented.

Marek Czerni *Representation theorems for solutions of a system of linear inequalities*

In the talk we present representation theorems for continuous solutions of a system of functional inequalities

$$\begin{cases} \psi[f(x)] \leq g(x)\psi(x), \\ (-1)^p\psi[f^2(x)] \leq (-1)^p g[f(x)]g(x)\psi(x) \end{cases} \tag{1}$$

where ψ is an unknown function, f, g are given functions, f^2 denotes the second iterate of f and $p \in \{0, 1\}$.

We assume the following hypotheses about the given functions f and g:

(H_1) The function $f\colon I \longrightarrow I$ is continuous and strictly increasing in an interval $I = [0, a|$ ($a > 0$ may belong to I or not). Moreover $0 < f(x) < x$ for $x \in I^\star = I \setminus \{0\}$, $f(I) = I$.

(H_2) The function $g\colon I \longrightarrow \mathbb{R}$ is continuous in I and $g(x) < 0$ for $x \in I$.

We shall be concerned with such solutions of (1) that for some fixed solution φ of a linear homogeneous functional equation

$$\varphi[f(x)] = g(x)\varphi(x)$$

or

$$\varphi[f(x)] = -g(x)\varphi(x)$$

the finite limit

$$\lim_{x\to 0^+} \frac{\psi(x)}{\varphi(x)}$$

exists.

Stefan Czerwik *Effective formulas for the Stirling numbers*

It is known that Stirling numbers play important role in many areas of mathematics and applications. We shall present some results about the Stirling numbers. We introduce new definition of the Stirling numbers of second kind. Moreover, we shall present some effective formulas for the Stirling numbers of the first kind.

Zoltán Daróczy *Nonconvexity and its application*

Joint work with Zsolt Páles.

Let $I \subset \mathbb{R}$ be a nonempty open interval. The following characterization of a continuous nonconvex function $f : I \longrightarrow \mathbb{R}$ is applicable for a number of questions in the theory of mean values.

THEOREM
Let $f\colon I \longrightarrow \mathbb{R}$ be a nonconvex continuous function on I. Then there exist $a \neq b$ in I such that

$$f(ta + (1-t)b) > tf(a) + (1-t)f(b)$$

holds for all $0 < t < 1$.

Judita Dascăl *On a functional equation with a symmetric component*

Let $I \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1,\ q \in (0,1)$, such that $r \neq q$, $r \neq \frac{1}{2}$ and $q \neq \frac{1}{2}$. In this presentation we give all the functions $f, g\colon I \longrightarrow \mathbb{R}_+$ such that

$$f\Big(\frac{x+y}{2}\Big)\,[r(1-q)g(y) - (1-r)qg(x)] = \frac{r-q}{1-2q}\,[(1-q)f(x)g(y) - qf(y)g(x)]$$

for all $x, y \in I$. Our main result is the following.

If the functions $f, g\colon I \longrightarrow \mathbb{R}_+$ are solutions of the above functional equation, then the following cases are possible:

(1) If $r \neq \frac{q^2}{q^2+(1-q)^2}$ and $r \neq \frac{q}{2q-1}$ then there exist constants $a, b \in \mathbb{R}_+$ such that

$$f(x) = a \quad \text{and} \quad g(x) = b \qquad \text{for all } x \in I;$$

(2) If $r = \frac{q^2}{q^2+(1-q)^2}$ then there exists an additive function $A\colon \mathbb{R} \longrightarrow \mathbb{R}$ and real numbers $c_1, c_2 > 0$ such that

$$g(x) = c_1 e^{A(x)} \quad \text{and} \quad f(x) = c_2 e^{2A(x)} \qquad \text{for all } x \in I;$$

(3) If $r = \frac{q}{2q-1}$ then there exist real numbers d_1, d_2, d_3 such that

$$g(x) = \frac{1}{d_1 x + d_2} > 0 \quad \text{and} \quad f(x) = d_3 \frac{1}{d_1 x + d_2} > 0 \qquad \text{for all } x, y \in I.$$

Conversely, the functions given in the above cases are solutions of the previous equation.

Joachim Domsta *An example of a group of commuting boosts*

In this talk a construction of a particular group of invariant linear maps for $\mathbb{R}^{1+3}$ is given. The set of mappings is the same as the one of the Lorentz group, but the group action is not simply the composition of maps. It is chosen in such a way, that the group of rotations is the same, as in the Lorentz group. But all boosts form a subgroup, which does not hold in the Lorentz group. Additionally, this subgroup is abelian. An interesting fact is, that the one dimensional subgroups of the boosts are simultaneously (one dimensional) subgroups of the Lorentz group.

Piotr Drygaś *Functional equations and effective conductivity in composite material with non perfect contact.*

We consider a conjugation problem for harmonic functions in multiply connected circular domains. This problem is rewritten in the form of the $\mathbb{R}$-linear boundary value problem by using equivalent functional-differential equations in a class of analytic functions. It is proven that the operator corresponding to the functional-differential equations is compact in the Hardy-type space. Moreover, these equations can be solved by the method of successive approximations under some natural conditions. This problem has applications in mechanics of composites when the contact between different materials is imperfect. It is given information about effective conductivity tensor with fixed accuracy for macroscopic isotropy composite material.

Włodzimierz Fechner *On some functional-differential inequalities related to the exponential mapping*

We examine some functional-differential inequalities which are related to the exponential function. In particular, we show that its solutions can be written as a product of the exponential function and a convex mapping. Our results are closely connected with the Hyers–Ulam stability of functional-differential equations and, in particular, with some of the results obtained in 1998 by Claudi Alsina and Roman Ger in [1].

[1] C. Alsina, R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373-380.

Roman Ger *On functional equations related to functional analysis – selected topics*

The talk focuses on the occurrence of various functional analysis aspects in the theory of functional equations and vice versa. Among others, the topics discussed concern representation theorems, generalizations of the Hahn–Banach type theorems and their geometric counterparts (separation results), characterizations of various kinds of Banach spaces, functional equations in Banach algebras, convex analysis, algebraic analysis, generalized polynomials, abstract orthogonalities and related equations, global isometries and their perturbations, stability and approximation theory and geometry of Banach spaces.

Dorota Głazowska *An invariance of the geometric mean in the class of Cauchy means*

We determine all the Cauchy conditionally homogeneous mean-type mappings for which the geometric mean is invariant, assuming that one of the generators of Cauchy mean is a power function.

Grzegorz Guzik *Derivations in some model of quantum gravity*

A sketch of a role of derivations, i.e., linear operators satisfying a Leibniz's rule in a new model of quantum gravity proposed by polish astrophysicist M. Heller and co-workers is presented. This promising model is an alternative to popular modern superstrings theories and it gives a hope to unification of relativity and quanta.

Konrad J. Heuvers *Some partial Cauchy difference equations for dimension two*

Let G be an abelian group and X a vector space over the rationals. For $\Phi\colon G \longrightarrow X$ its 1-st Cauchy difference is the function $K_2\Phi\colon G^2 \longrightarrow X$ defined by

$$K_2\Phi(x_1, x_2) := \Phi(x_1 + x_2) - \Phi(x_1) - \Phi(x_2)$$

and in general, for $n = 2, 3, \ldots$, the $(n-1)$-th Cauchy difference of Φ is the function $K_n\Phi\colon G^n \longrightarrow X$ defined by

$$K_n\Phi(x_1, \ldots, x_n) := \sum_{r=1}^{n} (-1)^{n-r} \sum_{|J|=r} \Phi(x_J)$$

where $\emptyset \neq J \subset I_n = \{1, \ldots, n\}$ and $x_J = \sum_{j \in J} x_j$. If $\Psi: G^n \longrightarrow X$, then its i-th partial difference of order r ($r = 2, 3, \ldots$), $K_r^{(i)}\Psi: G^{n+r-1} \longrightarrow X$, is its Cauchy difference of order r with respect to its i-th variable with all the others held fixed. For $n = 2$ and $i = 1, 2$ we have

$$K_2^{(1)}\Psi(x_1, x_2; x_3) = \Psi(x_1 + x_2, x_3) - \Psi(x_1, x_3) - \Psi(x_2, x_3)$$

and

$$K_2^{(2)}\Psi(x_1; x_2, x_3) = \Psi(x_1, x_2 + x_3) - \Psi(x_1, x_2) - \Psi(x_1, x_3).$$

In this talk the solutions of the following equations are given.

1. $K_2^{(1)} f_2 = K_2^{(2)} f_1$, where $f = \langle f_1, f_2 \rangle: G^2 \longrightarrow X^2$ (a 2-dim "curl"$= 0$).
2. $K_2^{(1)} f_1 + K_2^{(2)} f_2 = 0$, where $f = \langle f_1, f_2 \rangle: G^2 \longrightarrow X^2$ (a 2-dim "div"$= 0$).
3. $K_2^{(1)} f = K_2^{(2)} f$, where $f: G^2 \longrightarrow X$.
4. $K_2^{(1)} f = \lambda K_2^{(2)} f$, where $\lambda \neq 0, 1$ and $f: G^2 \longrightarrow X$. (Here the special case $\lambda = -i$ corresponds to a "Cauchy–Riemann equation".)

Eliza Jabłońska *On Christensen measurability and a generalized Gołąb–Schinzel equation*

Let X be a real linear space. We consider solutions $f: X \longrightarrow \mathbb{R}$ and $M: \mathbb{R} \longrightarrow \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y) = f(x)f(y) \qquad \text{for } x, y \in X, \tag{1}$$

where f is bounded on a Christensen measurable nonzero set as well as f is Christensen measurable. Our results refer to some results of C.G. Popa and J. Brzdęk.

Justyna Jarczyk *On an equation involving weighted quasi-arithmetic means*

We report on a progress made recently in studying solutions (φ, ψ) of the equation

$$\begin{aligned} \kappa x + (1-\kappa)y = \lambda\varphi^{-1}\left(\mu\varphi(x) + (1-\mu)\varphi(y)\right) \\ + (1-\lambda)\psi^{-1}\left(\nu\psi(x) + (1-\nu)\psi(y)\right), \end{aligned} \tag{1}$$

where $\kappa, \lambda \in \mathbb{R} \setminus \{0, 1\}$ and $\mu, \nu \in (0, 1)$. When $\kappa = \mu = \nu = \frac{1}{2}$ all twice continuously differentiable solutions of (1) were found by D. Głazowska, W. Jarczyk, and J. Matkowski. Later Z. Daróczy and Zs. Páles determined all continuously differentiable solutions of (1) in the case $\kappa = \mu = \nu$.

Witold Jarczyk *Iterability in a class of mean-type mappings*

Joint work with Janusz Matkowski.

Embeddability of a given pair of means in a continuous iteration semigroup of pairs of homogeneous symmetric strict means is considered.

Hans-Heinrich Kairies *On Artin type characterizations of the Gamma function*

E. Artin's monograph on the Gamma function contains two characterizations using the functional equation

$$f(x+1) = xf(x), \qquad x \in \mathbb{R}_+ \tag{F}$$

and the multiplication formula

$$f\Big(\frac{x}{p}\Big)f\Big(\frac{x+1}{p}\Big)\dots f\Big(\frac{x+p-1}{p}\Big) = (2\pi)^{\frac{1}{2}(p-1)}p^{\frac{1}{2}-x}f(x), \qquad x \in \mathbb{R}_+\,. \tag{M$_p$}$$

They read as follows.

THEOREM A
Assume that $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is continuously differentiable and satisfies* (F) *and* (M_p) *for some* $p \in \{2,3,4,\dots\}$. *Then* $f = \Gamma$.

THEOREM B
Assume that $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is continuous and satisfies* (F) *and* (M_p) *for every* $p \in \{2,3,4,\dots\}$. *Then* $f = \Gamma$.

We discuss both theorems with respect to their optimality.

Zygfryd Kominek *Stability of a quadratic functional equation on semigroups*

The stability problem of the functional equation of the form

$$f(x+2y) + f(x) = 2f(x+y) + 2f(y),$$

is investigated. We prove that if the norm of the difference between the left-hand side and the right-hand side of the equation is majorized by a function ω of two variables having some standard properties, then there exists a unique solution F of our equation and the norm of the difference between F and the given function f is controlled by a function depending on ω.

Krzysztof Król *Application of the least squares method and the decomposition method to solving functional equations*

In the talk we consider the approximate solution of the linear functional equation

$$y[f(x)] = g(x)y(x) + F(x), \tag{1}$$

in the class of continuous functions.

We use the least squares method for finding the approximate solution of the equation (1). In this method the accurate solution of the equation (1) may be approximated by the function

$$y_n(x) = \sum_{j=1}^{n} p_j \Phi_j(x),$$

where Φ_j , $j = 1, \ldots, n$, are given, continuous, linear independent functions, and coefficients p_j, $j = 1, \ldots, n$, are solutions of the system of equations

$$\sum_{j=1}^{n} p_j \int_a^b \Psi_i(x)\Psi_j(x)\,dx = \int_a^b \Psi_i(x)F(x)\,dx,$$

where $\Psi_i(x) = \Phi_i[f(x)] - g(x)\Phi_i(x)$ and $i = 1, \ldots, n$. We apply the least squares method to solving the exemplary equation.

Next we use the decomposition method for finding the approximate solution of the equation (1). At certain assumptions we show that the accurate solution of the equation (1) may be uniformly approximated by the function

$$y(x) = \sum_{n=0}^{\infty} \varphi_n(x),$$

where

$$\varphi_0(x) = -\frac{F(x)}{g(x)}, \quad \varphi_n(x) = \frac{\varphi_{n-1}(x)}{g(x)}, \qquad n = 1, 2, \ldots \ .$$

We prove that if there exists $0 \le \alpha < 1$ such that

$$\|\varphi_{n+1}\| \le \alpha \|\varphi_n\|, \qquad n = 0, 1, \ldots$$

then the series $\sum_{n=0}^{\infty} \varphi_n(x)$ is uniformly convergent to the accurate solution of the equation (1). Finally, we apply the decomposition method to solving the exemplary equation.

Arkadiusz Lisak *Some remarks on solutions of functional equations stemming from trapezoidal rule*

Joint work with Maciej Sablik.

The following functional equation (stemming from trapezoidal rule)

$$f_1(y) - g_1(x) = (y - x)[f_2(x) + f_3(sx + ty) + f_4(tx + sy) + f_5(y)]$$

with six unknown functions $g_1, f_i : \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, \ldots, 5$, where s and t are two fixed real parameters, has been solved by Prasanna K. Sahoo (University of Louisville, Louisville, USA). However, the solutions have been determined in particular for $s^2 \neq t^2$ (with $st \neq 0$) under high regularity assumptions on unknown functions (twice and four times differentiability). We solve this equation without any regularity assumptions on unknown functions for rational parameters s and t and with lesser regularity assumptions on unknown functions for real parameters s and t.

Fruzsina Mészáros *Functional equations stemming from probability theory*

Joint work with Károly Lajkó.

Special cases of the almost everywhere satisfied functional equation

$$g_1\left(\frac{x}{c(y)}\right)\frac{1}{c(y)}f_Y(y) = g_2\left(\frac{y}{d(x)}\right)\frac{1}{d(x)}f_X(x)$$

are investigated for the given positive functions c, d and unknown functions g_1, g_2, f_X and f_Y. This functional equation has important role in the characterization of distributions, whose conditionals belong to given scale families and have specified regressions.

Vladimir Mityushev *Application of functional equations to composites and to porous media*

Boundary value problems for multiply connected domains describe various physical phenomena in composites and porous media. One of the important constant of such problems constructed as a functional is the effective conductivity. Estimation of the effective conductivity can help to predict and to optimize properties of new created materials. It is shown that discussed boundary value problems can be effectively solved by reduction to iterative functional equations. New exact and approximate analytical formulae for the effective conductivity have been deduced. Further possible applications are discussed.

Takeshi Miura *A note on stability of Volterra type integral equation*

Let $\mathbb{R}$ be the real number field and let X be a complex Banach space. Suppose that p is a continuous function from $\mathbb{R}$ to the complex number field. The purpose of this talk is to give a sufficient condition in order that the equation

$$f(t) - f(0) = \int_0^t p(s)f(s)\,ds \qquad (\forall\, t \in \mathbb{R}) \tag{$*$}$$

has the stability in the sense of Hyers–Ulam: for every $\varepsilon \geq 0$ and continuous map $f\colon \mathbb{R} \longrightarrow X$ satisfying

$$\left\| f(t) - f(0) - \int_0^t p(s)f(s)\,ds \right\| \leq \varepsilon \qquad (\forall\, t \in \mathbb{R}),$$

there exists a solution $g\colon \mathbb{R} \longrightarrow X$ of the equation $(*)$ such that

$$\|f(t)-g(t)\| \le K\varepsilon \qquad (\forall\, t \in \mathbb{R}),$$

where K is a non-negative constant, depending only on the function p.

Janusz Morawiec *On the set of probability distribution solutions of a linear equation of infinite order*

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tau\colon \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ be a function which is strictly increasing and continuous with respect to the first variable, measurable with respect to the second variable. We are interested in the following problem: How much can we say about the class of all probability distribution solutions of the equation

$$F(x) = \int_\Omega F(\tau(x,\omega))\, dP(\omega)\,?$$

Jacek Mrowiec *On nonsymmetric t-convex functions*

Let $t \in (0,1)$ be a fixed number. It is known that if a function f defined on a convex domain D is t-convex, i.e., satisfies the condition

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y), \qquad x,y \in D, \tag{$*$}$$

then it is a midconvex (Jensen-convex) function, i.e., it satisfies the inequality

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2}$$

for all $x, y \in D$ (see [1] or [2], Lemma 1). Some years ago Zs. Páles has posed the following problem: Suppose that a function f satisfies the condition $(*)$ but only for $x<y$. Does this imply midconvexity of f? The partial answer to this question is given.

[1] N. Kuhn, *A note on t-convex functions*, General Inequalities **4**, Birkhäuser Verlag, 1984, 269-276.

[2] Z. Daróczy, Zs. Páles, *Convexity with given infinite weight sequences*, Stochastica **XI-1** (1987), 5-12.

Anna Mureńko *A generalization of the Gołąb-Schinzel functional equation*

We consider solutions $M, f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\circ : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of the functional equation

$$f(x+M(f(x))y) = f(x)\circ f(y),$$

under the following additional assumptions:

(a) f is continuous at a point;

(b) $M^{-1}(\{0\}) = \{0\}$;

(c) $\circ$ is commutative and associative.

Adam Najdecki *On stability of some functional equation*

Let $\mathcal{A}$ be a complex Banach algebra, S and T nonempty sets, and $h\colon T \longrightarrow \mathcal{A}$. Moreover, let $a_j \in \mathbb{C}$ and $g_j\colon S \times T \longrightarrow S$ for $j \in \mathbb{N}$. We are going to discuss the stability of the functional equation

$$\sum_{j=1}^{\infty} a_j f(g_j(s,t)) = h(t)f(s), \qquad s \in S,\ t \in T,$$

in the class of functions $f : S \longrightarrow \mathcal{A}$.

Andrzej Olbryś *On some functional inequality connected with t-Wright convexity and Jensen-convexity*

Let $t \in (0,1)$ be a fixed number, $L(t)$ – the smallest field containing the set $\{t\}$, and let X be a linear space over the field K, where $L(t) \subset K \subset R$. Let, moreover, $D \subset X$ be a $L(t)$-convex set, i.e., such set that $\alpha D + (1-\alpha)D \subset D$ for all $\alpha \in L(t) \cap (0,1)$.

In the talk we study connections between functions $f\colon D \longrightarrow \mathbb{R}$ satisfying the inequality

$$\frac{f(tx+(1-t)y) + f((1-t)x+ty)}{2} + f\Big(\frac{x+y}{2}\Big) \le f(x) + f(y), \qquad x, y \in D$$

and Jensen convex functions.

Boris Paneah *On the solvability of the identifying problem for general functional operators with linear arguments*

We start with a new problem for a general linear functional operator

$$(\mathcal{P}F)(x) = \sum_{j=1}^{N} c_j(x) F\big(a_j(x)\big), \qquad x \in D \subset \mathbb{R}^n,$$

where $F \in C(I,B)$, $I = [-1,1]$, B a Banach space, a_j and c_j given functions. This problem is intimately connected in some sense with approximation theory and can be described shortly as follows: find a finite-dimensional subspace $\mathcal{K} \subset C(I,B)$, a one-dimensional manifold $\Gamma \subset D$ and a subspace $C_{\langle\tau\rangle} = C_{\langle\tau\rangle}(I,B) \subset C(I,B)$ such that for an arbitrary $\varepsilon > 0$ the relation $|\mathcal{P}F|_{\langle\tau\rangle} < \varepsilon$ implies the inequality

$$\inf_{\varphi \in \mathcal{K}} |F - \varphi|_{\langle\tau\rangle} < c\varepsilon$$

with c a positive constant not depending on ε nor on F. If such a triple $(\mathcal{K},\ \Gamma,\ C_{\langle\tau\rangle})$ is found, we say that the identifying problem for the operator $\mathcal{P}$ is $(\Gamma,\ \mathcal{K})$ - solvable in the space $C_{\langle\tau\rangle}$. In particular, the well-known Hyers–Ulam result related to the functional Cauchy operator $\mathfrak{C}F = F(x,y) - F(x) - F(y)$ with $(x,y) \in \mathbb{R}^2$ can be reformulated as follows: the identifying problem for the operator $\mathfrak{C}$ is $(\mathbb{R}^2,\ \ker\mathfrak{C})$ – solvable in the space $C_{\langle\tau\rangle}$.

In the second part of the talk we give a solution of the identifying problem for a wide class of operators $\mathcal{P}$ with real c_j and linear functions a_j.

Boris Paneah *On the theory of the general linear functional operators with applications in analysis*

In the talk we discuss the recent results related to the solvability and qualitative properties of solutions of the general linear functional equations

$$\sum_{j=1}^{N} c_j(x)F\big(a_j(x)\big) = H(x),$$

where F are compact supported Banach-valued functions of a single variable and x are the points in a bounded domain $D \subset \mathbb{R}^n$, $n \geq 2$. When obtaining these results the new approach has been used. This is, first of all, the functional-analytic point of view which makes it possible to use the results and the methods of the classical functional analysis. Another novelty consists in systematical applying dynamical methods, based on the theory of the new dynamical systems introduced by the speaker (especially in connection with the problems in question). These results and methods will be considered at the first part of the talk. The second one is devoted to the (completely unexpected) connection of the above results with such divers fields of analysis as integral geometry, partial differential equations, and approximate solvability of the linear functional equations. The corresponding problems from these fields will be formulated (only some basic analysis is required for understanding) and their solutions will be given together with a list of unsolved problems (both in the theory of functional operators and in the applications).

Zsolt Páles *Comparison theorems in various classes of generalized quasi-arithmetic means*

Given a strictly increasing continuous function $f\colon I \longrightarrow \mathbb{R}$, the A_f quasi-arithmetic mean of the numbers $x_1,\dots,x_n \in I$ is defined by

$$A_f(x_1,\dots,x_n) = f^{-1}\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right).$$

The following classical result has attracted the attention of many researchers during the last decades.

Theorem

Let $f, g\colon I \longrightarrow \mathbb{R}$ be continuous strictly increasing functions. Then the following conditions are equivalent:

— *For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$,*

$$A_f(x_1, \dots, x_n) \le A_g(x_1, \dots, x_n);$$

— *for all $p \in I$ there exists $\delta > 0$ such that, for all $x, y \in]p-\delta, p+\delta[$,*

$$A_f(x, y) \le A_g(x, y);$$

— *$g \circ f^{-1}$ is convex;*

— *there exists a function $h : I \longrightarrow \mathbb{R}$ such that, for all $x, y \in I$,*

$$f(x) - f(y) \le h(y)\big(g(x) - g(y)\big);$$

— *if f, g are twice differentiable with $f'g' \neq 0$ then, for all $x \in I$,*

$$\frac{f''(x)}{f'(x)} \le \frac{g''(x)}{g'(x)}.$$

Our aim is to survey several extensions and of the above theorem related to various generalizations of quasi-arithmetic means.

Magdalena Piszczek *On a multivalued second order differential problem with Jensen multifunctions*

Let K be a closed convex cone with a nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F_t : t \ge 0\}$ is a regular cosine family of continuous Jensen set-valued functions $F_t\colon K \longrightarrow cc(K)$, $x \in F_t(x)$ for $t \ge 0$, $x \in K$ and $F_t \circ F_s = F_s \circ F_t$ for $s, t \ge 0$, then such family is twice differentiable and

$$DF_t(x)|_{t=0} = \{0\}, \qquad D^2F_t(x) = A_t(A(x) + D)$$

for $x \in K$ and $t \ge 0$, where $DF_t(x)$ denotes the Hukuhara derivative of $F_t(x)$ with respect to t, $\{A_t : t \ge 0\}$ is a regular cosine family of continuous additive multifunctions, $D \in cc(K)$ and $A(x) = D^2A_t(x)|_{t=0}$.

This result is a motivation for studying the existence and uniqueness of a solution

$$\Phi\colon [0, +\infty) \times K \longrightarrow cc(K),$$

which is Jensen with respect to the second variable, of the following differentiable problem

$$\begin{aligned} \Phi(0,x) &= \Psi(x), \\ D\Phi(t,x)|_{t=0} &= \{0\}, \\ D^2\Phi(t,x) &= A_\Phi(t,H(x)), \end{aligned}$$

where $H, \Psi\colon K \longrightarrow cc(K)$ are given continuous Jensen multifunctions, $D\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t and A_Φ is the additive, with respect to the second variable, part of Φ.

Vladimir Protasov *Self-similarity equations in L_p spaces*

We consider functional difference equations with linear contractions of the argument (self-similarity equations). Let $L_p[0,1]$ be the space of vector-functions from the segment $[0,1]$ to $\mathbb{R}^d$ with the norm $\|v\|_p = \bigl(\int_0^1 |v(t)|^p\,dt\bigr)^{\frac{1}{p}}$. Suppose we have an arbitrary family of affine operators $\{\tilde{A}_1,\ldots,\tilde{A}_m\}$ in $\mathbb{R}^d$. We always assume this family to be irreducible (there is no common invariant affine subspace, different from the whole $\mathbb{R}^d$). Let us also have a partition of the segment $[0,1]$ with nodes $0 = b_0 < \ldots < b_m = 1$. We denote $\Delta_k = [b_{k-1}, b_k]$, $r_k = b_k - b_{k-1}$. The affine function $g_k(t) = tb_k + (1-t)b_{k-1}$ maps $[0,1]$ to the segment Δ_k. The *self-similarity operator* $\tilde{\mathbf{A}}$:

$$\bigl[\,\tilde{\mathbf{A}}\,v\,\bigr](t) \;=\; \tilde{A}_k\, v\bigl(g_k^{-1}(t)\bigr), \qquad t \in \Delta_k,\ k = 1,\ldots,m,$$

is defined on $L_1[0,1]$. The equation $\tilde{\mathbf{A}}v = v$ is called *self-similarity equation.* Special cases of such equations are applied in the ergodic theory, wavelets theory, approximation theory, probability, etc. Most of the classical fractal curves (such as Cantor singular function, Koch and de Rham curve, etc.) are solution of suitable self-similarity equations. Refinement equations from wavelets theory and approximation subdivision algorithms are also actually self-similarity equations.

We consider the following problem: what are the conditions on the operators $\{\tilde{A}_k\}$ and on the partition of the segment $[0,1]$ necessary and sufficient for the self-similarity equation to possess an L_p-solution? What can be said about the uniqueness and regularity of the solutions?

We derive a sharp criterion of solvability for these equations in the spaces L_p and C, compute the exponents of regularity and estimate the moduli of continuity. We show that the solution is always unique, whenever exists. The answers are given in terms of the so-called p-radius of the family of operators $\{\tilde{A}_k\}$. This, in particular, gives a geometric interpretation of the p-radius in terms of spectral radii of certain operators in the space $L_p[0,1]$.

Viorel Radu *The fixed point method to generalized stability of functional equations in normed and random normed spaces*

D.H. Hyers in 1941 gave an affirmative answer to a question of S.M. Ulam, concerning the stability of group homomorphisms, for Banach spaces: *Let E_1*

and E_2 *be Banach spaces and* $f\colon E_1 \longrightarrow E_2$ *be such a mapping that*

$$\|f(x+y) - f(x) - f(y)\| \le \delta, \tag{1}$$

for all $x, y \in E_1$ *and a* $\delta > 0$*, that is* f *is* $\delta-$additive. *Then there exists a unique* additive $T\colon E_1 \longrightarrow E_2$*, which satisfies* $\|f(x) - T(x)\| \le \delta$, $x \in E_1$. *In fact,*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \qquad x \in E_1, \tag{Hyers}$$

T. Aoki, D. Bourgin and Th.M. Rassias studied the stability problem with unbounded Cauchy differences: it is supposed that $||\mathcal{D}f(x,y)|| \le \delta(x,y)$ and the *stability estimations* are of the form $||f(x) - S(x)|| \le \varepsilon(x)$, where S is a solution, that is, it *verifies the functional equation* $\mathcal{D}S(x,y) = 0$, and for $\varepsilon(x)$ explicit formulae are given, which depend on the control δ as well as on the equation.

We discuss the generalized Ulam–Hyers stability for functional equations in abstract spaces and show how the stability results can be obtained by a fixed point method, initiated in (Radu [4], 2003) and developed in (Cădariu & Radu [2], 2004) as well as in subsequent papers.

[1] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure and Appl. Math. **4(1)** (2003), Art.4 (http://jipam.vu.edu.au).

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed points approach*, Grazer Math. Ber. **346** (2004), 323-350.

[3] L. Cădariu, V. Radu, *Fixed points method for the stability of some functional equations*, Carpathian J. Math. **23**, No. 1-2, (2007), 63-72.

[4] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, Cluj-Napoca **IV(1)** (2003), 91-96.

Ewa Rak *Distributivity between uninorms and nullnorms*

The problem of distributivity has been posed many years ago (cf. Aczel [1], pp. 318-319). A new direction of investigations is mainly concerned of distributivity between triangular norms and triangular conorms ([5] p. 17). Recently, many authors have dealt with solution of distributivity equation for aggregation functions ([3]), fuzzy implications ([2], [10]), uninorms and nullnorms ([6], [7], [8], [9]), which are generalization of triangular norms and conorms.

Our consideration was motivated by intention of determining algebraic structures which have weaker assumptions than uninorms and nullnorms. In particular, the assumption of associativity is not necessary in consideration of distributivity equation. Moreover, if we omit commutativity assumption, consideration of the left and right distributivity conditions is reasonable. A characterization of such binary operations is interesting not only from a theoretical point of view, but also for their applications, since they have proved to be useful in several fields like fuzzy logic framework, expert system, neural networks or fuzzy quantifiers (cf. [4]).

Previous results about distributivity between uninorms and nullnorms can be obtained as simple corollaries.

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Acad. Press, New York, 1966.

[2] M. Baczyński, *On a class of distributive fuzzy implications*, Internat. J. Uncertainty, Fuzzines Knowledge-Based Syst. **9** (2001), 229-238.

[3] T. Calvo, *On some solutions of the distributivity equation*, Fuzzy Sets and Systems **104** (1999), 85-96.

[4] J. Drewniak, P. Drygaś, E. Rak, *Distributivity equations for uninorms and nullnorms*, Fuzzy Sets and Systems **159** (2008), 1646-1657.

[5] J. Fodor, M. Roubens, *Fuzzy Preference Modeling and Multicriteria Decision Support*, Kluwer Acad. Publ., New York, 1994.

[6] M. Mas, G. Mayor, J. Torrens, *The distributivity condition for uninorms and t-operators*, Fuzzy Sets and Systems **128** (2002), 209-225.

[7] E. Rak, *Distributivity equation for nullnorms*, J. Electrical Engin. **56**, 12/s (2005), 53-55.

[8] E. Rak, P. Drygaś, *Distributivity between uninorms*, J. Electrical Engin. **57**, 7/s (2006), 35-38.

[9] D. Ruiz, J. Torrens, *Distributive idempotent uninorms*, Internat. J. Uncertainty, Fuzzines Knowledge-Based Syst. **11** (2003), 413-428.

[10] D. Ruiz, J. Torrens, *Distributivity of residual implications over conjunctive and disjunctive uninorms*, Fuzzy Sets and Systems **158** (2007), 23-37.

Themistocles M. Rassias *On some major trends in mathematics*

In this talk I shall attempt to present some ideas regarding the present state and the near future of mathematics. Since assessments and any predictions in this field of science are necessarily subjective, I shall communicate to you the opinions of renowned contemporary mathematicians with some of whom I have recently come into contact. I will include of course the significant contribution of Polish mathematicians.

Themistocles M. Rassias *New and old problems in mathematical analysis*

We present some new and old problems that are inspired by D. Hilbert problems [Göttinger Nachrichten (1900), 253-297, and the Bull. Amer. Math. Soc. **8** (1902), 437-479] and S. Smale problems [Mathematics: Frontiers and Perspectives, Mathematical Problems for the Next Century, International Mathematical Union, Amer. Math. Soc., 2000].

In particular emphasis is given to problems related to the representation of functions in several variables by means of functions of a smaller number of variables (J. d'Alembert, V. Arnold, N. Kolmogorov), A.D. Aleksandrov problem for isometric mappings and S.M. Ulam problem for approximate homomorphisms.

The interaction between analysis and geometry is discussed through old and new results, examples and further questions for future work.

[1] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications vol. **34**, Birkhäuser, Boston – Basel – Berlin, 1998.

[2] Th.M. Rassias, J. Šimša, *Finite Sums Decompositions in Mathematical Analysis*, John Wiley & Sons, Wiley-Interscience Series in Pure and Applied Mathematics, Chichester, 1995.

[3] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[4] G. Isac, Th.M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. & Math. Sci. **19**(2) (1996), 219-228.

[5] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Applicandae Mathematicae **62**(1) (2000), 23-130.

Maciej Sablik *Generalized homogeneity of some means*

We deal with means $M\colon I \times I \longrightarrow I$ which are $\circ$-homogenous, i.e., satisfying the equation

$$M(s \circ x, s \circ y) = s \circ M(x, y)$$

for all $s, x, y \in I$, where $\circ$ is a binary operation defined on $I \times I$. In particular, given a quasiarithmetic mean, we determine all continuous, associative and commutative operations $\circ$ with respect to which the mean is homogeneous. Also, we characterize given quasiarithmetic means as homogeneous with respect to a couple of suitable operations. This is a generalization of the well known result on characterization of the arithemtic mean as the only one which is homogeneous both with respect to ordinary multiplication and addition (see eg. J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989).

The results have been partially obtained in collaboration with Małgorzata Pałys.

Ekaterina Shulman *Some extensions of the Levi–Civita equation*

Let T be a representation of a topological group G on a Banach space X. A vector x is called *finite* if there is a finite dimensional subspace $M \subset X$ such that $T_g x \in M$ for each $g \in G$. A finite dimensional subspace $L \subset X$ is called *special* if there is a finite dimensional subspace $M \subset X$ such that $T_g L \bigcap M \neq 0$, for each $g \in G$. We prove that a subspace is special if and only if it contains a finite vector. Using this result we describe continuous solutions $f_j(x)$ of the functional equation

$$\sum_{j=1}^{m} a_j(x) f_j(x+y) = \sum_{i=1}^{n} u_i(x) v_i(y)$$

which extends the well known Levi–Civita equation

$$f(x+y) = \sum_{i=1}^{n} u_i(x) v_i(y).$$

Justyna Sikorska *On a conditional exponential functional equation and its stability*

Joint work with Janusz Brzdęk.

We study a conditional functional equation of the form

$$\gamma(x+y) = \gamma(x-y) \implies f(x+y) = f(x)f(y) \qquad (*)$$

for a given function γ. Condition $(*)$ with $\gamma = \|\cdot\|$ is the so called isosceles orthogonally exponential functional equation. We show the form of the solutions and investigate the stability of the presented equation. Moreover, we study the pexiderized version of $(*)$.

Barbara Sobek *Pexider equation on a restricted domain*

Let $(X,+)$ be a uniquely 2-divisible Abelian topological group which has a base $\mathcal{B}$ of open neighbourhoods of 0 satisfying the following conditions:

(a) if $B \in \mathcal{B}$ and $x \in B$, then $\frac{x}{2} \in B$,

(b) if $B \in \mathcal{B}$ and $x \in X$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $\frac{x}{2^n} \in B$.

Assume that U is a nonempty, open and connected subset of $X \times X$. Let

$$U_1 := \{x : \ (x,y) \in U \text{ for some } y \in X\},$$
$$U_2 := \{y : \ (x,y) \in U \text{ for some } x \in X\}$$

and

$$U_+ := \{x+y : \ (x,y) \in U\}.$$

We consider the Pexider functional equation

$$f(x+y) = g(x) + h(y) \qquad \text{for } (x,y) \in U,$$

where $f\colon U_+ \longrightarrow K$, $g\colon U_1 \longrightarrow K$ and $h\colon U_2 \longrightarrow K$ are unknown functions and $(K,+)$ is an Abelian group. In particular, we improve Theorem 1 in [F. Radó, J.A. Baker, *Pexider's equation and aggregation of allocations*, Aequationes Math. **32** (1987), 227-239].

Paweł Solarz *Some iterative roots for homeomorphisms with periodic points*

Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism such that $\operatorname{Per} F$, the set of all periodic points of F, is nonempty. It is known that there is an integer $n > 1$ such that

$$\operatorname{Per} F = \{z \in S^1 : \ F^n(z) = z \text{ and } \forall_{0<k<n}\ F^k(z) \neq z\}.$$

If $\operatorname{Per} F \neq S^1$, the equation

$$G^m(z) = F(z), \qquad z \in S^1,$$

where $m \geq 2$, may not have continuous and orientation-preserving solutions. However, if $\gcd(m, n) = 1$, then there are infinitely many such solutions having periodic points of period n. These solutions depend on an arbitrary function. We give the general construction of these solutions.

Tomasz Szostok *On an equation connected to Lobatto quadrature rule*

Joint work with Barbara Koclęga-Kulpa.

Quadrature rules are used in numerical analysis for estimating integrals by the following formula

$$\int_x^y f(t)\,dt \ \approx\ (y-x)\sum_{i=1}^{n} \alpha_i f(a_i x + (1-a_i)y)$$

where the error term depends on the derivative of f. Further for the polynomials of certain degree (depending on the length and form of the quadrature considered) the above formula is exact. This means that polynomials satisfy equations of the type

$$F(y) - F(x) = (y-x)\sum_{i=1}^{n} \alpha_i f(a_i x + (1-a_i)y)$$

where F is the primitive function of f. In the current talk we solve an equation of this type with the right-hand side containing two endpoints and two other points from the interval $[x, y]$ which are symmetric with respect to the midpoint of this interval. Thus we deal with the equation

$$F(y) - F(x) = (y-x)[\alpha f(x) + \beta f(ax + (1-a)y) + \beta f((1-a)x + ay) + \alpha f(y)]$$

where functions $f, F\colon \mathbb{R} \longrightarrow \mathbb{R}$ and constants $\alpha, \beta, a \in \mathbb{R}$ are unknown.

Jacek Tabor *Extensions of conditionally convex functions*

Joint work with Józef Tabor.

Let $V \subset \mathbb{R}^N$ be a closed bounded convex set and let $f\colon \partial V \longrightarrow \mathbb{R}$ be a continuous conditionally convex function, that is

$$f(\alpha x+(1-\alpha)y)\le \alpha f(x)+(1-\alpha)f(y)\quad \text{for } \alpha\in[0,1],\ x,y\in V:[x,y]\subset V,$$

where $[x,y]=\{\alpha x+(1-\alpha)y:\ \alpha\in[0,1]\}$. Then there exists a continuous convex function $F\colon V\longrightarrow\mathbb{R}$ such that $F|_{\partial V}=f$.

We also show that the assumption that V is bounded is essential.

Józef Tabor *Generalized approximate midconvexity*

Joint work with Jacek Tabor.

Let X be a normed space and let $V\subset X$ be an open convex set. Let $\alpha\colon[0,\infty)\longrightarrow\mathbb{R}$ be a given nondecreasing function. A function $f\colon V\longrightarrow\mathbb{R}$ is $\alpha(\cdot)$-midconvex if

$$f\Big(\frac{x+y}{2}\Big)\le\frac{f(x)+f(y)}{2}+\alpha(\|x-y\|)\qquad\text{for all } x,y\in V.$$

We prove that if f is $\alpha(\cdot)$-midconvex and locally bounded at a point then

$$f(rx+(1-r)y)\le rf(x)+(1-r)f(y)+P_\alpha(r,\|x-y\|)$$

for $x,y\in V$, $r\in[0,1]$, where $P_\alpha\colon[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is a function depending on α. Three different estimations of P_α are considered.

Aleksej Turnšek *Mappings approximately preserving orthogonality*

We present some results on orthogonality preserving and approximately orthogonality preserving mappings in the setting of inner product C^*-modules (Hilbert spaces). Some open questions are also considered.

Jian Wang *The relation between isometric and affine operators on F^*-spaces*

In this talk, we study the relation between isometries and affine operators on F^*-spaces, showing that for AL_β-spaces $(0<\beta\le1)$ E and F if E possesses a normalized complete disjoint atoms system $\{e_\gamma\}_{\gamma\in\Gamma}$, then an isometric embedding $T\colon E\longrightarrow F$ with $T\emptyset=\emptyset$ is linear if and only if, for any $\gamma\in\Gamma$,

(i) $P_\gamma(TE)\subseteq\operatorname{span}(Te_\gamma)$ when $0<\beta<1$, and

(ii) $P_\gamma(TE)\subseteq\operatorname{span}(Te_\gamma)$ and $T(-e_\gamma)\in\operatorname{span}(Te_\gamma)$ when $\beta=1$,

where P_γ is a principal band projection from F onto B_{Te_γ}. At the same time, we prove also that every onto isometry $T\colon(s)_p\to(s)_p$ (resp., $l^{(p_n)}\longrightarrow l^{(p_n)}$, in particular, $l_\beta(\Gamma)\longrightarrow l_\beta(\Gamma)$) is affine. For a number of results for isometric mappings one may see works of M. Day, Ding and Huang, and Th.M. Rassias.

Szymon Wąsowicz *On some inequalities between quadrature operators*

In the class of 3-convex functions we establish the order structure of the set of six well known operators connected with an approximate integration:

two-point and three-point Gauss–Legendre quadratures, Chebyshev quadrature, four-point and five-point Lobatto quadratures and the Simpson's Rule. We show that 12 (of 15 possible) inequalities are true while only 3 fail. For 5-convex functions the situation diametrally differs: only 3 inequalities hold and 12 fail. Among the considered inequalities at least one seems to be not trivial. To prove it we use some method connected with the spline approximation of convex functions of higher order.

Wirginia Wyrobek *Measurable orthogonally additive functions modulo a discrete subgroup*

Joint work with Tomasz Kochanek.

Under appropriate conditions on Abelian topological groups G and H, an orthogonality $\perp\subset G^2$ and a σ-algebra $\mathfrak{M}$ of subsets of G we decompose an $\mathfrak{M}$-measurable function $f\colon G\longrightarrow H$ which is orthogonally additive modulo a discrete subgroup K of H into its continuous additive and continuous quadratic part (modulo K).

Problems and Remarks

1. Remark. *On application of the multiplication formula to $\frac{1}{n}$-stable probability distributions*

From the multiplication formula for the gamma function we obtain for $n\in\mathbb{N}$, $x>0$, that

$$\frac{n\Gamma(nx)}{\Gamma(x)}=(n^n)^x\cdot\frac{\Gamma(x+\frac{1}{n})}{\Gamma(\frac{1}{n})}\cdot\frac{\Gamma(x+\frac{2}{n})}{\Gamma(\frac{2}{n})}\cdot\ldots\cdot\frac{\Gamma(x+\frac{n-1}{n})}{\Gamma(\frac{n-1}{n})}. \tag{1}$$

If $\xi_{\frac{1}{n}},\xi_{\frac{2}{n}},\ldots,\xi_{\frac{n-1}{n}}$, are independent random variables with $\Gamma(\frac{1}{n},1),\ldots,\Gamma(\frac{n-1}{n},1)$ probability distributions, respectively, then for the right hand side we can write

$$RHS_{(1)}(n,x)=E(n^n\cdot\xi_{\frac{1}{n}}\cdot\xi_{\frac{2}{n}}\cdot\ldots\cdot\xi_{\frac{n-1}{n}})^x, \tag{2}$$

where E stands for the expectation.

On the other hand we can write

$$LHS_{(1)}(n,x)=E(\sigma_{1/n})^{-x}, \tag{3}$$

if $\sigma_{1/n}$ has the strictly stable probability distribution defined by its Laplace transform

$$E\left(e^{-s\sigma_{1/n}}\right)=e^{(-s^{\frac{1}{n}})},\qquad \operatorname{Re}s\geq 0.$$

Indeed, by Fubini's theorem we obtain

$$E(\sigma_{1/n})^{-x}=\int_0^\infty v^{-x}\,dP_{\sigma_{\frac{1}{n}}}(v)$$

$$= \int_0^\infty \left(\int_0^\infty y^{x-1} e^{-vy} dy \right) \frac{1}{\Gamma(x)} dP_{\sigma_{\frac{1}{n}}}(v)$$

$$= \frac{1}{\Gamma(x)} \int_0^\infty y^{x-1} e^{-y^{\frac{1}{n}}} dy$$

$$= n\frac{\Gamma(nx)}{\Gamma(x)}, \qquad x > 0.$$

Thus, by the uniqueness theorem for the inverse two-sided Laplace transform, from (1)-(3) we obtain the equality of distributions

$$\sigma_{1/n} \stackrel{d}{=} \frac{1}{n^n \cdot \xi_{1/n} \cdot \xi_{2/n} \cdot \ldots \cdot \xi_{(n-1)/n}}, \qquad n = 1, 2, 3, \ldots. \tag{4}$$

For $n = 2$ it is known as a property of the $\Gamma(\frac{1}{2}, 1)$ probability distribution (P. Lévy).

Joachim Domsta

2. Problem. *Lipschitz perturbation of continuous linear functionals*

Let X be a normed space, $D \subseteq X$ be an open convex set and let $f\colon D \longrightarrow \mathbb{R}$ be a Lipschitz perturbation of a linear functional, i.e., let f be of the form

$$f = x^* + \ell,$$

where x^* is a continuous linear functional and $\ell\colon D \longrightarrow \mathbb{R}$ is an ε-Lipschitz function, i.e.,

$$|\ell(x) - \ell(y)| \leq \varepsilon\|x - y\| \qquad (x, y \in D).$$

Then, for $x, y \in D$ and $t \in [0, 1]$, we have

$$\begin{aligned} &\big|tf(x) + (1-t)f(y) - f(tx + (1-t)y)\big| \\ &\quad = \big|t\ell(x) + (1-t)\ell(y) - \ell(tx + (1-t)y)\big| \\ &\quad \leq t\big|\ell(x) - \ell(tx + (1-t)y)\big| + (1-t)\big|\ell(y) - \ell(tx + (1-t)y)\big| \\ &\quad \leq t\varepsilon\|x - (tx + (1-t)y)\| + (1-t)\varepsilon\|y - (tx + (1-t)y)\| \\ &\quad = 2\varepsilon t(1-t)\|x - y\|. \end{aligned}$$

On the other hand, in the case $X = \mathbb{R}$, we have the following converse of the above observation.

Claim

Let I be an open interval and $\varepsilon \geq 0$. Assume that $f\colon I \longrightarrow \mathbb{R}$ satisfies, for all $x, y \in I$ and $t \in [0, 1]$, the inequality

$$\big|tf(x) + (1-t)f(y) - f(tx + (1-t)y)\big| \leq 2\varepsilon t(1-t)|x - y|. \tag{1}$$

Then there exists a constant $c \in \mathbb{R}$ such that the function $\ell\colon I \longrightarrow \mathbb{R}$ defined by $\ell(x) := f(x) - cx$ is (2ε)-Lipschitz.

The proof is elementary and is left to the reader. However, the following more general and open problem seems to be of interest.

PROBLEM
Does there exist a constant γ (that may depend on X and D) such that, whenever a function $f\colon D \longrightarrow X$ satisfies inequality (1) for all $x, y \in D$ and $t \in [0,1]$, then there exists a continuous linear functional x^* such that the function $\ell := f - x^*$ is $\gamma\varepsilon$-Lipschitz on D?

Zsolt Páles

3. Problems. 1. Find all mappings $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the following functional equation is satisfied:

$$\|T\vec{u} \times T\vec{v}\| = \|\vec{u} \times \vec{v}\|, \qquad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^3.$$

Geometrically the problem is asking for the determination of all mappings $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which preserve area of parallelograms in the Euclidean 3-dimensional space.

2. Find all mappings $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the following functional equation is satisfied

$$|(T\vec{u} \times T\vec{v}) \cdot T\vec{w}| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \qquad \text{for all } \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3.$$

Geometrically the problem is asking for the determination of all mappings $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, which preserve volume of parallelepipeds in the Euclidean 3-dimensional space.

Note. In the above two problems the symbols $\times, \cdot$ denote vector product and dot (scalar) product, respectively.

Remark. It will be interesting to formulate and solve the analogous functional equations for mappings $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ for the determination of all mappings which preserve area (resp. volume) of the surfaces of balls (resp. solid balls) in the Euclidean 3-dimensional space. The same problem remains open for ellipsoids in $\mathbb{R}^3$.

3. Examine whether there exists a mapping $f\colon B \longrightarrow \mathbb{R}^3$, that is at the same time *harmonic* as well as *homeomorphism.*

Remark. $f\colon B \longrightarrow \mathbb{R}^3$ is a harmonic mapping if its three coordinate real-valued functions on the unit ball B are harmonic functions, i.e., if each one of the three coordinate functions of f satisfies the Laplace equation in B. The mapping $f\colon B \longrightarrow \mathbb{R}^3$ is a homeomorphism if it is a bijective and bicontinuous mapping.

The same problem remains open for the n-dimensional case where $n = 4, 5, \ldots$.

Themistocles M. Rassias

4. Remark. *Some remarks on the talk of T. Miura*

Recently the number of papers whose title includes the word combination "Ulam stability" threateningly grows. Little by little these works begin to involve operators which are very far from the linear functional operators $\mathcal{P}$ in several variables (in the framework of these operators this notion have appeared). The interest to the type of stability in question is easily explained by its evident connection with the very important problem of the approximate solvability of the inequality $\|\mathcal{P}F\| < \varepsilon$. But to the late days any progress in the solvability of this problem in the class of functional equations in several variables was connected mainly with a success in guessing new types of the operators $\mathcal{P}$ to which it is possible (after a series of substitutions and arithmetic transformations) to apply the original construction of Hyers.

As to the classical operators of analysis (integral, differential, partial differential, etc.) to which from time to time some authors turn in order to cross them with the Ulam stability, for these operators the "Ulam problem" is successfully solving under different names during 70 years in the framework of functional analysis (Banach, Riesz, Schauder, Leray and others). I'll demonstrate this on the basis of the Miura's talk "A note on stability of Volterra type integral equation". The speaker delivered the following result.

> *Let $f\colon \mathbb{R} \longrightarrow B$ be a $C(\mathbb{R}, B)$-function, B a Banach space and*
>
> $$(Tf)(t) = f(0) + \alpha(t) \int_0^t p(s,t) f(s)\, ds$$
>
> *with α and p being continuous maps to $\mathbb{C}$. Then there is a function $\varphi\colon \mathbb{R} \longrightarrow B$ depending on f such that $\varphi - T\varphi = 0$ and $\|f-\varphi\|_C < m\varepsilon$ if $\|f - Tf\| \le \varepsilon$ with m a constant depending neither on f nor ε.*

(As a matter of fact, the speaker dealt with the simplest case $\alpha = 1$, $p(s,t) = p(s)$.) From the point of view of functional analysis this is a standard exercise related to the invertibility of linear operators in B-space. No hint at stability! The following solution does not require any comment. Denote by E the identical operators in $C(I,B)$ with I – a compact interval in $\mathbb{R}$, and let $f \in C(I,B)$. Then

1. the operator T is compact in $C(I,B) \Rightarrow$
2. the range of the operator $E - T$ is closed $\Rightarrow$
3. the *a priori* estimate

   $$\inf_{\varphi \in \ker(E-T)} |f - \varphi|_{C(I,B)} \le m|(E-T)f|_{C(I,B)}, \qquad f \in C(I,B)$$

   holds.

This completes the solution.

In the case $\alpha = 1$, $p(s,t) = p(s)$ the space $\ker(E - T)$ is one-dimensional and consists of the functions $\varphi = \lambda \exp(\int_0^t p(s)\,ds)$ with $\lambda \in B$.

In my opinion, the majority of results related to integro-differential operators with the reference to the stability, as a matter of fact, has the same nature: some usual property of an inverse operator is treated as the Ulam stability. But from this point of view any classical result in the theory of boundary problems for partial differential equations (the unique solvability of the Dirichlet problem for the Laplace operator, for example) may be treated as Ulam stability.

Boris Paneah

5. Remark. *Some remarks on the talk of Z. Kominek*

In his talk Prof. Z. Kominek considered the operator

$$\mathcal{P}\colon f(t) \longrightarrow f(x+2y) + f(x) - 2f(x+y) - 2f(y)$$

from $C(\mathbb{R})$ to $C(\mathbb{R}^2)$ and formulated the following proposition: there is a function $w(x,y)$ such that if $|(\mathcal{P}f)(x,y)| < |w(x,y)|$ for all points $(x,y) \in \mathbb{R}^2$ then the equation $\mathcal{P}F = 0$ is uniquely solvable, and for some function ψ the relation $|f - F| \le \psi(w)$ holds. No information about w and ψ had been mentioned.

It is easily seen that the above operator $\mathcal{P}\colon C(I) \longrightarrow C(D)$ with $D = \{(x,y) \mid \ x + 2y \le 1,\ x \ge 0,\ y \ge 0\}$ and $I = [0,1]$ satisfies all conditions formulated in my talk and providing solvability of the identifying problem for $\mathcal{P}$. According to the main result of this talk, if $|(\mathcal{P}f)(x,y)|_{\langle 2\rangle} < \varepsilon$ for all points (x,y) of the straight line $\Gamma = \{(x,y) \mid x = \frac{1}{3}t,\ y\frac{1}{3}t;\ 0 \le t \le 1\}$, then the inequality $|f(t) - \lambda t^2|_{\langle 2\rangle} < c\varepsilon$ holds for a constant λ and all points t, $0 \le t \le 1$. The constant c does not depend on f nor t, $|\cdot|_{\langle 2\rangle}$ is the norm in the space $C_{\langle 2\rangle}$ of continuous in I functions satisfying the 2-Hölder condition at $t = 0$. What is important here is that the initial condition of the smallness of $\mathcal{P}f$ is imposed only at points of an one-dimensional manifold Γ and the approximate solution f of the relation $|\mathcal{P}_\Gamma f| < \varepsilon$ is close not to the subspace $\ker \mathcal{P}$, but to the subspace $\ker \mathcal{P}_\Gamma = \{\varphi \mid \ \varphi(3t) - 2\varphi(2t) - \varphi(t) = 0,\ 0 \le t \le 1\}$, where $\mathcal{P}_\Gamma$ denotes the restriction of the operator $\mathcal{P}$ to Γ.

Boris Paneah

6. Remark. *On functional equations "of Kuczma's type"*

The first paper on functional equations written by Marek Kuczma (1935-1991) had appeared 50 years ago. Together with his colleagues and students he developed the theory of functional equations called "in a single variable" or "iterative" – later on.

Having this anniversary in mind I proposed to introduce in the title of our paper [1] the name "functional equation of Kuczma's type".

But, motivated by what have been said at the Conference on names assigned to stability problems, I have found this idea was not good. First of all, the late

Marek Kuczma himself would be against it. And all who knew him personally would confirm this prediction. Moreover, the new name is unprecise, may led to confusions, and the existing ones are satisfactory.

The aim of this remark is to declare that we decided to change the title of our paper, as indicated in [1].

[1] B. Choczewski, M. Czerni, *Special solutions of a linear functional equation of Kuczma's type*. New title: *Special solutions of a linear iterative functional equation*, Aequationes Math., to appear.

Bogdan Choczewski

7. Problem. *A functional equation with two complex variables*

The functional equation

$$\varphi(z + 2\pi i) = \varphi(z), \qquad z \in \mathbb{C} \tag{1}$$

in the class of entire functions has the general solution of the form $\varphi(z) = \psi(\exp z)$, where ψ is an arbitrary entire function. Equation (1) characterizes the complex exponent.

Let $f(z)$ be a given polynomial or entire function. Consider now the functional equation

$$\varphi[z + 2\pi i, f(z)] = \varphi[z, f(z)], \qquad z \in \mathbb{C}, \tag{2}$$

where $\varphi(z, w)$ is unknown entire function with respect to z and to w.

Conjecture

The general solution of (2) *has the form* $\varphi(z, w) = \psi(\exp z, w)$*, where* ψ *is an arbitrary entire function of two variables.*

It is worth noting that an artificial insert of the exponent in φ does not solve the problem. For instance, the function $\psi_0(u, w) = \ln u$ produces $\varphi_0(z, w) = \ln \exp z = z$ in the strip $0 \leq \operatorname{Im} z \leq 2\pi$ periodically continued onto $\mathbb{C}$. The function φ_0 satisfies (2), however, φ_0 and ψ_0 are not entire functions.

One can see also that the functional equation $\varphi(z + 2\pi i, w) = \varphi(z, w)$, $(z, w) \in \mathbb{C}^2$ (compare to equation (1)) has only exponent in z solutions. But the restriction $w = f(z)$ in (2) yields complications.

The case $f(z) = z$ and its application to Arnold's problem [1, p. 168-170] of topologically elementary functions were discussed in [2].

[1] V.I. Arnold (ed.), *Arnold's Problems*, Springer, Berlin, 2004.

[2] V. Mityushev, *Exponent in one of the variables*, Jan Długosz University of Częstochowa, Scientific Issues, Mathematics XII, Częstochowa, 2007.

Vladimir Mityushev

8. Problems. *Stability of the orthogonality preserving property and related problems*

1. As it was reminded in Prof. Aleksej Turnšek's talk, the orthogonality preserving property for linear mappings between Hilbert spaces is stable. Namely (cf. [1], [4]), if $f: X \longrightarrow Y$ is a *linear* mapping satisfying

$$x \perp y \implies fx \perp^{\varepsilon} fy, \qquad x, y \in X$$

(where $u \perp^{\varepsilon} v$ means that $|\langle u|v\rangle| \leq \varepsilon \|u\| \, \|v\|$), then there exists a *linear* mapping $g: X \longrightarrow Y$ satisfying

$$x \perp y \implies gx \perp gy, \qquad x, y \in X$$

and such that

$$\|fx - gx\| \leq \left(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right) \cdot \min\{\|fx\|, \|gx\|\}, \qquad x \in X.$$

Problem 1. The question is whether the linearity can be omitted in the above statement (both in assumption and in assertion).

2. Orthogonality preserving property can also be defined for mappings between normed spaces, with one of the possible notions of orthogonality. Attempting to solve the stability problem for this property, with respect to the *isosceles orthogonality* ($u \perp v \Leftrightarrow \|u+v\| = \|u-v\|$) I encountered the following problem concerning the stability of isometries.

Problem 2. Let $f: X \longrightarrow Y$ be a linear mapping between Banach spaces satisfying

$$|\, \|fx - fy\| - \|x - y\| \,| \leq \varepsilon \|x - y\|, \qquad x, y \in X.$$

Does there exists a linear isometry $I: X \longrightarrow Y$ such that

$$\|fx - Ix\| \leq \delta(\varepsilon)\|x\|, \qquad x \in X$$

(with some $\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\lim_{\varepsilon \longrightarrow 0^+} \delta(\varepsilon) = 0$)?

Without the linearity assumption, the question has a negative answer, as it was shown by G. Dolinar [3, Proposition 4].

Yet during the meeting Problem 2 has been solved by Prof. Vladimir Protasov. For finite-dimensional spaces the answer to the question is positive (some compactness argument is sufficient) whereas for infinite ones it is generally not true. Namely, for an increasing sequence of positive numbers $\{\alpha_k\}_{k \in \mathbb{N}}$ such that the series $\sum_{k \in \mathbb{N}} (1 - \alpha_k^2)$ converges, consider the following norm in a Hilbert space l_2:

$$\|x\|_\alpha := \sup\left\{\|x\|_{l_2}, \left|\frac{x_1}{\alpha_1}\right|, \left|\frac{x_2}{\alpha_2}\right|, \dots\right\}, \qquad x = (x_1, x_2, \dots) \in l_2 .$$

Denote the space l_2 endowed with the new norm $\|\cdot\|_\alpha$ by H_α (this space is reflexive). Now, let $A_k\colon H_\alpha \to H_\alpha$ be an operator interchanging the kth and $(k+1)$st coordinates. It can be shown that A_k is an approximate isometry (with given ε provided that k is sufficiently big) but it cannot be approximated by a linear isometry, as the only linear isometries on the considered space are coordinate symmetries (i.e., $Te_i = \pm e_i$, $i = 1, 2, \dots$).

Afterwards, A. Turnšek pointed out that the problem has been already considered in the literature, e.g. in [2].

[1] J. Chmieliński, *Stability of the orthogonality preserving property in finite-dimensional inner product spaces*, J. Math. Anal. Appl. **318** (2006), 433-443.

[2] G.G. Ding, *The approximation problem of almost isometric operators by isometric operators*, Acta Math. Sci. (English Ed.), **8** (1988), 361-372.

[3] G. Dolinar, *Generalized stability of isometries*, J. Math. Anal. Appl. **242** (2000), 39-56.

[4] A. Turnšek, *On mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **336** (2007), 625-631.

Jacek Chmieliński

List of Participants

ADAM Marcin, Wyższa Szkoła Zarządzania Ochroną Pracy, ul. Bankowa 8, 40-007 KATOWICE, Poland, e-mail: madam@wszop.edu.pl

ADAMEK Mirosław, Katedra Matematyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: madamek@ath.bielsko.pl

BAHYRYCZ Anna, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: bah@ap.krakow.pl

BARON Karol, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: baron@us.edu.pl

BÁLINTNÉ DASCAL Judita, Department of Mathematics, University of Debrecen, DEBRECEN, Hungary, e-mail: jdascal@math.klte.hu

BRZDĘK Janusz, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jbrzdek@ap.krakow.pl

CĂDARIU-BRĂILOIU Liviu, Department of Mathematics, “Politehnica" University of Timişoara, Piaţa Victoriei, No. 2, 300006 TIMIŞOARA, Romania, e-mail: liviu.cadariu@mat.upt.ro, lcadariu@yahoo.com

CHMIELIŃSKI Jacek, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jacek@ap.krakow.pl

CHOCZEWSKI Bogdan, Wydział Matematyki Stosowanej, Akademia Górniczo-Hutnicza, al. Mickiewicza 30, 30-059 KRAKÓW, Poland, e-mail: smchocze@cyf-kr.edu.pl, smchocze@cyfronet.pl

CHUDZIAK Jacek, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: chudziak@univ.rzeszow.pl

CZERNI Marek, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: mczerni@ap.krakow.pl

CZERWIK Stefan, Instytut Matematyki, Politechnika Śląska, ul. Kaszubska 23, 44-100 GLIWICE, Poland, e-mail: stefan.czerwik@polsl.pl

DARÓCZY Zoltán, Institute of Mathematics, University of Debrecen, Pf. 12, 4010 DEBRECEN, Hungary, e-mail: daroczy@math.klte.hu

DOMSTA Joachim, Wydział Fizyki Technicznej i Matematyki Stosowanej, Politechnika Gdańska, ul. Narutowicza 11/12, 80-952 GDAŃSK, Poland, e-mail: jdomsta@mif.pg.gda.pl

DRYGAŚ Piotr, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16a, 35-959 RZESZÓW, Poland, e-mail: drygaspi@univ.rzeszow.pl

FECHNER Włodzimierz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: fechner@math.us.edu.pl

GER Roman, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: romanger@us.edu.pl

GŁAZOWSKA Dorota, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland, e-mail: D.Glazowska@wmie.uz.zgora.pl

GUZIK Grzegorz, Wydział Matematyki Stosowanej, Akademia Górniczo-Hutnicza, al. Mickiewicza 30, 30-059 KRAKÓW, Poland, e-mail: guzikgrz@wms.mat.agh.edu.pl

HEUVERS Konrad J., Michigan Technical University, 219 Trinklein St., FRANKENMUTH, MI 48734-1513, USA, e-mail: kheuvers@mtu.edu

JABŁOŃSKA Eliza, Katedra Matematyki, Politechnika Rzeszowska, W. Pola 2, 35-959 RZESZÓW, Poland, e-mail: elizapie@prz.rzeszow.pl

JARCZYK Justyna, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland, e-mail: j.jarczyk@wmie.uz.zgora.pl

JARCZYK Witold, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland, e-mail: w.jarczyk@wmie.uz.zgora.pl

KAIRIES Hans-Heinrich, Institut für Mathematik, Technische Universität Clausthal, Erzstrasse 1, 38678 CLAUSTHAL–ZELLERFELD 1, Germany, e-mail: kairies@math.tu-clausthal.de

KOMINEK Zygfryd, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: zkominek@ux2.math.us.edu.pl

KRÓL Krzysztof, Instytut Matematyki, Politechnika Śląska, ul. Kaszubska 23, 44-100 GLIWICE, Poland, e-mail: Krzysztof.Krol@polsl.pl

LISAK Arkadiusz, Wyższa Szkoła Informatyki i Zarządzania, ul. Sucharskiego 2, 35-225 RZESZÓW, Poland, e-mail: alisak@wsiz.rzeszow.pl

MÉSZÁROS Fruzsina, Institute of Mathematics, University of Debrecen, Pf. 12, 4010 DEBRECEN, Hungary, e-mail: mefru@math.klte.hu

MITYUSHEV Vladimir, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: vmityu@yahoo.com

MIURA Takeshi, Department of Applied Mathematics and Physics, Graduate Yamagata University, YONEZAWA 992-8510, Japan, e-mail: miura@yz.yamagata-u.ac.jp

MORAWIEC Janusz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: morawiec@math.us.edu.pl

MROWIEC Jacek, Katedra Matematyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: jmrowiec@ath.bielsko.pl

MUREŃKO Anna, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: aniam@univ.rzeszow.pl

NAJDECKI Adam, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: najdecki@univ.rzeszow.pl

OLBRYŚ Andrzej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: andrzej.olbrys@wp.pl

PANEAH Boris, Department of Mathematics, Technion, 32000 HAIFA, Israel, e-mail: peter@tx.technion.ac.il

PÁLES Zsolt, Institute of Mathematics, University of Debrecen, Pf. 12, 4010 DEBRECEN, Hungary, e-mail: pales@math.klte.hu

PIETRZYK Andrzej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland

PISZCZEK Magdalena, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: magdap@ap.krakow.pl

PROTASOV Vladimir, Department of Mechanics and Mathematics, Moscow State University, Vorobyovy Gory, 119992 MOSCOW, Russia, e-mail: v-protassov@yandex.ru

RADU Viorel, Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, Bv. Vasile Pârvan 4, RO-300223 TIMIŞOARA, Romania, e-mail: radu@math.uvt.ro

RAK Ewa, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 a, 35-959 RZESZÓW, Poland, e-mail: ewarak@univ.rzeszow.pl

RASSIAS Themistocles M., Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 ATHENS, Greece, e-mail: trassias@math.ntua.gr

SABLIK Maciej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: mssablik@us.edu.pl

SCHLEIERMACHER Adolf, Rablstr. 18/V, 81669 MÜNCHEN, Germany, e-mail: adsle@aol.com

SHULMAN Ekaterina, Department of Mathematocal Analysis, Vologda State Pedagogical University, 6 Orlova Str., VOLOGDA, 160000, Russia, e-mail: shulmank@yahoo.com

SIKORSKA Justyna, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: sikorska@math.us.edu.pl

SOBEK Barbara, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: b_sobek@wp.pl

SOLARZ Paweł, Instytut Matematyki, Akademia Pedagogiczna, Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: psolarz@ap.krakow.pl

SZCZAWIŃSKA Joanna, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jszczaw@ap.krakow.pl

SZOSTOK Tomasz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: szostok@ux2.math.us.edu.pl

TABOR Jacek, Instytut Matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 KRAKÓW, Poland, e-mail: tabor@im.uj.edu.pl, Jacek.Tabor@im.uj.edu.pl

TABOR Józef, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: tabor@univ.rzeszow.pl

TURNŠEK Aleksej, Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, 1000 LJUBLJANA, Slovenia, e-mail: aleksej.turnsek@fmf.uni-lj.si

WANG Jian, Department of Mathematics, Fujian Normal University, Fazhou, FUJIAN 350007, P. R. China, e-mail: wangjian@fjnu.edu.cn

WĄSOWICZ Szymon, Katedra Matematyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: swasowicz@ath.bielsko.pl

WYROBEK Wirginia, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: wwyrobek@math.us.edu.pl

projekt graficzny Jadwiga Burek

ISSN 1643–6555

Redakcja/Dział Promocji
Wydawnictwo Naukowe UP
30-084 Kraków, ul. Podchorążych 2
tel./fax (012) 662–63–83, tel. (012) 662–67–56
e-mail: wydawnictwo@up.krakow.pl

Zapraszamy na stronę internetową:
http://www.wydawnictwoup.pl

druk i oprawa
Zespół Poligraficzny UP, zam. 2/09

ISSN 1643-6555