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Sign-changing Lyapunov functions in linear extensions of dynamical systems

Abstract. In this note we consider sets of linear extensions of dynamical systems on a torus. We examine regularity of the systems by means of a given sign-changing Lyapunov function. The main result of the paper is to give conditions of regularity for the set of differential equations with degenerated matrix of coefficients.

Let us consider a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ \dfrac{dy}{dt} = A(x)y, \end{cases} \tag{1}$$

where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_n)$, $A(x)$ is a square, n-dimensional matrix, which elements are periodic with period 2π, continuous with respect to each variable x_j, $j = 1, \dots, k$, it means it is specified on an k-dimensional torus T_k. The set of all such functions which are continuous and periodic with period 2π with respect to each variable x_j, $j = 1, \dots, k$, is denoted by $C^0(T_k)$. We assume that the function $f(x)$ satisfies the Lipschitz inequality $||f(x) - f(\bar{x})|| \le L||x - \bar{x}||$ for all $x, \bar{x} \in T_k$, $L = \mathrm{const} > 0$, where $||y||^2 = \langle y, y\rangle$ is the Euclidean norm in the space $\mathbb{R}^n$, $y \in \mathbb{R}^n$, and $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ is an inner product in $\mathbb{R}^n$. We denote by $C_{Lip}(T_k)$ a space of functions $f(x) \in C^0(T_k)$, which satisfy the Lipschitz inequality. It follows that $f(x) \in C_{Lip}(T_k)$ and $A(x) \in C^0(T_k)$. Let us also denote by $||A|| = \max\limits_{||y||=1} ||Ay||$ the norm of a $n\times n$-dimensional matrix A taken as an operator. In $C^0(T_k)$ we distinguish a subspace $C'(T_k; f)$ of functions $F(x)$ such that the superposition $F(x(t; x))$ is continuously differentiable with respect to $t \in \mathbb{R}$, where $x(t; x)$ is a solution to the Cauchy problem

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$$\frac{dx}{dt} = f(x), \qquad x\big|_{t=0}\,, \ \forall x \in T_k\,.$$

We define

$$\dot{F}(x) \stackrel{\text{df}}{=} \frac{dF(x(t;x))}{dt}\bigg|_{t=0} \qquad \text{for } \dot{F}(x) \in C^0(T_k).$$

In $C^0(T_k)$ we also distinguish a subspace $C^1(T_k)$ of functions $F(x)$, which have continuous first derivatives with respect to each variable x_j, $j = 1, \dots, k$. Let $u(\varphi) \in C^1(T_k)$. Then $\dot{u}(\varphi) = \sum_{j=1}^{k} \frac{\partial u(\varphi)}{\partial \varphi} f_j(\varphi) = \frac{\partial u}{\partial \varphi} f(\varphi)$.

Definition 1
We say that the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ \dfrac{dy}{dt} = A(x)y + h(x), \qquad h(x) \in C^0(T_k) \end{cases} \tag{2}$$

possesses a torus

$$y = u(x),$$

if $u(x) \in C'(T_k; f)$ and the identity

$$\dot{u}(x) \equiv A(x)u(x) + h(x), \qquad \forall x \in T_k$$

holds.

Example
Let us consider a system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = 1, \\ \dfrac{dx_2}{dt} = \sqrt{3}, \\ \dfrac{dy}{dt} = (\gamma - \sin(x_1 + x_2))y + h(x_1, x_2), \end{cases}$$

where $\gamma = \text{const} \in \mathbb{R}$, $h(x_1, x_2) \in C^1(T_2)$. An invariant torus for the system has the following form.

I. Case $\gamma > 0$.

$$\begin{aligned} y &= u(x_1, x_2) \\ &= -\int_0^{\infty} e^{-\gamma\tau - \frac{1}{1+\sqrt{3}}[\cos((1+\sqrt{3})\tau + x_1 + x_2) - \cos(x_1+x_2)]} h(\tau + x_1, \sqrt{3}\tau + x_2)\, d\tau. \end{aligned}$$

II. Case $\gamma < 0$.

$$y = u(x_1, x_2) = \int\limits_{-\infty}^{0} e^{\gamma\tau + \frac{1}{1+\sqrt{3}}[\cos((1+\sqrt{3})\tau + x_1 + x_2) - \cos(x_1+x_2)]} h(\tau + x_1, \sqrt{3}\tau + x_2)\, d\tau.$$

III. Case $\gamma = 0$. The invariant torus for the system fails to exists for every $h(x) \in C^1(T_2)$. For example, when $h \equiv 2$, the torus does not exist.

Definition 2
Let $C(x)$ be an $n{\times}n$-dimensional continuous matrix, $C(x) \in C^0(T_k)$. Then the function $G_0(\tau, x)$ defined by

$$G_0(\tau, x) = \begin{cases} \Omega^0_\tau(x) C(x(\tau, x)), & \tau \le 0, \\ \Omega^0_\tau(x)[C(x(\tau, x)) - I_n], & \tau > 0, \end{cases} \tag{3}$$

which satisfies the estimate

$$||G_0(\tau, x)|| \le K e^{-\gamma|x|}, \tag{4}$$

where K and γ are positive constants, is called the *Green function* of the invariant torus for the system (1).

$\Omega^t_x(x)$ is the fundamental matrix of the solutions of the system $\frac{dy}{dt} = A(x(t; x))y$ which takes the value of the n-dimensional identity matrix for $t = x$ $\Omega^t_x(x)\big|_{t=x} = I_n$.

If the Green function (3) exists, then for every vector function $h(x) \in C^0(T_k)$ an invariant torus for the system (2) exists and it is defined by the formula

$$y = u(x) = \int\limits_{-\infty}^{\infty} G_0(\tau, x) h(x(\tau, x))\, d\tau.$$

Definition 3
We say that the system (1) is *regular* if there exists a unique Green function (3) satisfying (4).

It is obvious [3], that the system (1) is regular when the square form

$$V = \langle S_0(x)y, y\rangle\,, \tag{5}$$

with the symmetric matrix $S_0(x) \in C^1(T_k)$, exists and its derivative along the solutions of the system (1) is positive definite:

$$\dot{V} = \left\langle \left[\frac{\partial S_0(x)}{\partial x} f(x) + S_0(x)A(x) + A^T(x)S_0(x)\right] y, y \right\rangle \ge \varepsilon ||y||^2, \tag{6}$$

$\varepsilon = \text{const} > 0$, and the matrix $S_0(x)$ satisfies the condition

$$\det S_0(x) \ne 0, \qquad \forall x \in T_k\,.$$

Dealing with problems of regularity of systems we find out issues which have not been touched upon in researches. We shall prove the existence of the regular system (1) for which $\det A(x) \equiv 0$ for all $x \in T_k$. The next problem is the analysis of right-hand sides of the system (1) for the which the derivative of the square form (5) along the solutions of the system is positive definite.

First of all, let us notice that the inequality (6) does not change for small perturbations of the vector function $f(x)$ and the matrix $A(x)$. We will show that in the right-hand side of the system (1), $f(x)$ can be substituted by any different function $b(x) \in C_{Lip}(T_k)$ and at the same time the matrix $A(x)$ can be chosen in such a way that the derivative of the square form (5) along the solutions of the system is positive definite. Thus the matrix $A(x)$ has the form

$$A(x) = S_0^{-1}(x)\left[B(x) + M(x) - 0.5\frac{\partial S_0(x)}{\partial x}b(x)\right], \tag{7}$$

where $B(x), M(x) \in C^0(T_k)$ are any matrices which satisfy

$$B^T(x) \equiv B(x), \qquad \langle B(x)x, x\rangle \geq \lambda||x||^2,\ \lambda = \text{const} > 0, \tag{8}$$

$$M^T(x) \equiv -M(x). \tag{9}$$

Let us check whether it is true. We consider the left-hand side of (6), substituting the function $f(x)$ with any vector function $b(x)$. We also assume the form of the matrix $A(x)$ to be like the one in (7):

$$\begin{aligned}\frac{\partial S_0(x)}{\partial x}f(x) + S_0(x)A(x) + A^T(x)S_0(x) = {} & \frac{\partial S_0(x)}{\partial x}b(x) + B(x) + M(x) \\ & - 0.5\,\frac{\partial S_0(x)}{\partial x}b(x) + B(x) + M^T(x) \\ & - 0.5\,\frac{\partial S_0(x)}{\partial x}b(x) \\ = {} & 2B(x).\end{aligned}$$

It follows that, when (8) is fulfilled, the inequality (6) is fulfilled for $\varepsilon = 2\lambda$. Then we get the following lemma.

Lemma

To any non-degenerate matrix $S_0(x) \in C^1(T_k)$ there corresponds the set of regular systems

$$\begin{cases} \dfrac{dx}{dt} = b(x), \\ \dfrac{dy}{dt} = S_0^{-1}(x)\left[B(x) + M(x) - 0.5\dfrac{\partial S_0(x)}{\partial x}b(x)\right]y, \end{cases} \tag{10}$$

where $b(x)$ is any vector function, $b(x) \in C_{Lip}(T_k)$, $B(x), M(x)$ are any continuous matrices satisfying (8) *and* (9)*.*

Remark 1
The derivative of the square form (5) with the symmetric non-degenerate matrix $S_0(x) \in C^1(T_k)$ along the solutions of the system (10) has the form $\dot{V} = 2\langle By, y\rangle$.

Theorem 1
Systems (1)*, in which* $\det A(x) \equiv 0$ *for all* $x \in T_k$ *, exist.*

Proof. We define

$$S_0(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}, \qquad \psi = x_1 + x_2 + x_3 + \ldots + x_k\,. \tag{11}$$

We shall consider the system (10) of the form

$$\begin{cases} \dfrac{dx_i}{dt} = \omega_i\,, \\ \dfrac{dy}{dt} = S_0^{-1}(\psi)\Big[B + M - \dfrac{1}{2}\cdot\dfrac{dS_0(\psi)}{d\psi}\displaystyle\sum_{j=1}^{k}\omega_j\Big]y, \end{cases} \qquad \omega_i = \text{const}, \tag{12}$$

where $i = 1, \ldots, k$ and $y \in \mathbb{R}^2$. Let B and M be constant matrixes

$$B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad b > 0, \qquad M = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}. \tag{13}$$

The derivative of the non-degenerate square form $V = y_1^2\cos\psi + 2y_1y_2 sin\psi - y_2^2\cos\psi$ along the solutions of the system (12) is positive definite, hence the system (12) is regular. Taking $\omega = \sum\omega_j$ we obtain

$$A(x) = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}\left\{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} - 0.5\omega\begin{pmatrix} -\sin\psi & \cos\psi \\ \cos\psi & \sin\psi \end{pmatrix}\right\}$$

$$= \begin{pmatrix} b\cos\psi + m\sin\psi & b\sin\psi - m\cos\psi - 0.5\omega \\ b\sin\psi - m\cos\psi + 0.5\omega & -b\cos\psi - m\sin\psi \end{pmatrix}. \tag{14}$$

We have $\det A(x) = -b^2 - m^2 + 0.25\omega^2$. Therefore, the identity $\det A(x) \equiv 0$ holds when $\omega^2 = 4(b^2 + m^2)$.

Remark 2
Using change of variables

$$y = \begin{pmatrix} \cos\frac{\psi}{2} & \sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & -\cos\frac{\psi}{2} \end{pmatrix} z, \qquad z \in \mathbb{R}^2,$$

the system (12) with matrices (11) and (13) can be transformed into the following system with constant coefficients

$$\begin{cases} \dfrac{dx_i}{dt} = \omega_i\,, \\ \dfrac{dz_1}{dt} = bz_1 + mz_2\,, \\ \dfrac{dz_2}{dt} = mz_1 - bz_2\,. \end{cases}$$

Remark 2 confirms once again that the system (1) is a regular one, because eigenvalues λ_i of the matrix of coefficients $\begin{pmatrix} b & m \\ m & -b \end{pmatrix}$ for the given system satisfy the condition $\operatorname{Re}\lambda_i = \lambda_i \neq 0$.

Remark 3

If the Green function (3) with the estimate (4) exists, then the function

$$G_t(\tau, x) = \begin{cases} \Omega^t_\tau C(x(\tau, x)), & \tau \le t, \\ \Omega^t_\tau [C(x(\tau, x)) - I_n], & \tau > t \end{cases}$$

is called (cf. [3]) the Green function of the problem of the bounded solutions of the system $\frac{dy}{dt} = A(x(t;x))y$. It means that for any function $h(t)$, which is continuous and bounded, the system

$$\frac{dy}{dt} = A(x(t;x))y + h(t), \qquad \forall x \in T_k$$

has the unique bounded solution

$$y = \int\limits_{-\infty}^{\infty} G_t(\tau, x)h(\tau)\,d\tau.$$

Based on the previous considerations let us note, that the linear system which corresponds to the system (12) with the matrix $A(2\omega t)$ given by (14) after replacing ω by 2ω can be written in the form

$$\begin{cases} \dot y_1 = (b\cos 2\omega t + m\sin 2\omega t)y_1 + (-m\cos 2\omega t + b\sin 2\omega t - \omega)y_2\,, \\ \dot y_2 = (-m\cos 2\omega t + b\sin 2\omega t + \omega)y_1 + (-b\cos 2\omega t - m\sin 2\omega t)y_2 \end{cases} \tag{15}$$

with constants $\omega, m, b \in \mathbb{R}$, $(b \neq 0)$. Let us note that $\det A(2\omega t) \equiv 0$ when the condition

$$\omega^2 = b^2 + m^2 \tag{16}$$

holds. The derivative of the non-degenerate square form $V = y_1^2 \cos\omega t + 2y_1y_2 \sin\omega t - y_2^2 \cos\omega t$ along the solutions of the system is positive definite, thus the system is exponentially dichotomous in $\mathbb{R}$ (cf. [1], [3]). It means that the non-homogenous system

$$\begin{cases} \dot{y}_1 = (b\cos 2\omega t + m\sin 2\omega t)y_1 + (-m\cos 2\omega t + b\sin 2\omega t - \omega)y_2 \\ \qquad + h_1(t), \\ \dot{y}_2 = (-m\cos 2\omega t + b\sin 2\omega t + \omega)y_1 + (-b\cos 2\omega t - m\sin 2\omega t)y_2 \\ \qquad + h_2(t) \end{cases} \tag{17}$$

has a unique bounded solution in $\mathbb{R}$ for any vector function $h(t)$ which is continuous and bounded in $\mathbb{R}$.

Since we want to write down the solution of the system (17), we simplify the system (15). On the basis of Remark 2 we use the change of variables

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ \sin\omega t & -\cos\omega t \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and the system (15) results in the system with constant coefficients

$$\begin{cases} \dot{z}_1 = bz_1 + mz_2\,, \\ \dot{z}_2 = mz_1 - bz_2\,. \end{cases} \tag{18}$$

Then another change of variables in the system (18) can be used:

$$z = Tr,$$

where

$$T = \begin{cases} \begin{pmatrix} \omega + b & -m \\ m & \omega + b \end{pmatrix}, & b > 0, \\ \begin{pmatrix} m & b - \omega \\ \omega - b & m \end{pmatrix}, & b < 0 \end{cases}$$

and we obtain the system with separated variables

$$\begin{cases} \dot{r}_1 = \omega r_1\,, \\ \dot{r}_2 = \ -\omega r_2\,. \end{cases}$$

Therefore, the bounded solution of the system (17) has the form

$$y = y^*(t) = L(t)T \int\limits_{-\infty}^{\infty} G(t,\tau)T^{-1}L^{-1}(\tau)h(\tau)\,d\tau,$$

where

$$G(t,\tau) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \exp\{-\omega(t-\tau)\}, & \tau \le t, \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \exp\{\omega(t-\tau)\}, & \tau > t, \end{cases}$$

$$L(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ \sin\omega t & -\cos\omega t \end{pmatrix}.$$

Under the condition (16) the system (17) can be transformed into the scalar equation

$$\begin{aligned} &\left(\frac{dy_1}{dt} - h_1(t)\right)[m\cos(2\omega t) - b\sin(2\omega t) - \omega] \\ &\qquad + \left(\frac{dy_2}{dt} - h_2(t)\right)[b\cos(2\omega t) + m\sin(2\omega t)] = 0. \end{aligned} \tag{19}$$

Remark 4
The equation (19) can be transformed into the form

$$\left(\frac{dy_1}{dt} - \bar{h}_1(t)\right)\sin t + \left(\frac{dy_2}{dt} - \bar{h}_2(t)\right)\cos t = 0, \tag{20}$$

where $\bar{h}_i(t) = \frac{1}{\omega} h_i\left(\frac{1}{\omega}t - \frac{\Delta+\pi}{2\omega}\right)$, $i = 1, 2$, $\cos\Delta = \frac{m}{\omega}$, $\sin\Delta = \frac{m}{\omega}$.

Now we consider (20) as a separate equation and we obtain that, apart from the solution $y = y^*(t) = (y_1^*(t), y_2^*(t))$, there exists the whole set of bounded solutions.

Remark 5
In the system (10) the variable $\xi \in T_l$ can be added to the variable x. Then we consider the system

$$\begin{cases} \dfrac{dx}{dt} = b(x,\xi), \\ \dfrac{d\xi}{dt} = \bar{b}(x,\xi), \\ \dfrac{dy}{dt} = S_0^{-1}\left[B(x,\xi) + M(x,\xi) - 0.5\dfrac{\partial S_0(x)}{\partial x} b(x,\xi)\right] y, \end{cases} \tag{21}$$

where $b(x,\xi), \bar{b}(x,\xi) \in C_{Lip}(T_{k+l})$ are any vector functions and for matrices $B(x,\xi), M(x,\xi) \in C^0(T_k \times T_l)$ identities $B^T \equiv B$, $M^T \equiv -M$ hold. The derivative of the square form (5) along the solutions of (21) has the form $\dot{V} = 2\langle B(x,\xi)y, y\rangle$.

Remark 6
If the derivative of the non-degenerate square form (5) along the solutions of the system

$$\begin{cases} \dfrac{dx}{dt} = b(x,\xi), \\ \dfrac{d\xi}{dt} = \bar{b}(x,\xi), \\ \dfrac{dy}{dt} = P(x,\xi)y, \end{cases}$$

where $x \in T_k$, $\xi \in T_l$, is positive definite, then

$$P(x,\xi) = S_0^{-1}(x)\left[B(x,\xi) + M(x,\xi) - 0.5\frac{\partial S_0(x)}{\partial x}b(x,\xi)\right], \tag{22}$$

where the matrix $B(x,\xi)$ is symmetric positive definite and the matrix $M(x,\xi)$ is skew-symmetric.

If the derivative of the square form (5) along the solutions of the system (21) is positive definite, then

$$\dot{V} = \left\langle \left[\frac{\partial S_0(x)}{\partial x}B(x,\xi) + S_0(x)P(x,\xi) + P^T(x,\xi)S_0(x)\right]y, y\right\rangle \geq \varepsilon||y||^2,$$

$\varepsilon = \text{const} > 0$. Let us take matrices B and M of the following forms:

$$B(x,\xi) = 0.5\left[\frac{\partial S_0(x)}{\partial x}B(x,\xi) + S_0(x)P(x,\xi) + P^T(x,\xi)S_0(x)\right], \tag{23}$$

$$M(x,\xi) = 0.5\left[S_0(x)P(x,\xi) - P^T(x,\xi)S_0(x)\right]. \tag{24}$$

Then, above-mentioned matrixes (23) and (24) satisfy conditions (8) and (9) ($\lambda = \frac{\varepsilon}{2}$) and the equality (22) also holds.

Remark 7
If for the non-degenerate square form (5) we consider the set of systems (1), where the derivative of the form along the solutions of these systems is positive definite, we are able only to increase the number of variables x. Decreasing the number of variables x is not always possible.

This is confirmed by the following example. We consider the matrix $S_0(\psi)$ of the form (11). Let $\tilde{x} = (x_2, \ldots, x_k)$, $x = (x_1, \tilde{x})$. Let us assume that functions $f_j(\tilde{x})$ with respect to $\tilde{x}$, $f_j(\tilde{x}) \in C_{Lip}(T_{k-1})$, $j = 2, \ldots, k$ exist and, moreover, matrices $A(\tilde{x})$ for which inequalities

$$\left\langle \left[\sum_{j=2}^{k}\frac{\partial S_0(\psi)}{\partial x_j}f_j(\tilde{x}) + S_0(\psi)A(\tilde{x}) + A^T(\tilde{x})S_0(\psi)\right]y, y\right\rangle \geq \varepsilon||y||^2, \tag{25}$$

$$\varepsilon = \text{const} > 0,\ \psi = x_1 + x_2 + x_3 + \ldots + x_k\,.$$

hold, exist. Taking into account the identities

$$S_0(\psi)\big|_{x_1=0} \equiv -S_0(\psi)\big|_{x_1=\pi},$$
$$\frac{\partial S_0(\psi)}{\partial x_j}\bigg|_{x_1=0} \equiv -\frac{\partial S_0(\psi)}{\partial x_j}\bigg|_{x_1=\pi} \qquad j=2,\ldots,k,$$

we obtain a contradiction to the formula (25):

$$\Big\langle\Big[\sum_{j=2}^{k}\frac{\partial S_0(\psi)}{\partial x_j}f_j(\tilde{x})+S_0(\psi)A(\tilde{x})+A^T(\tilde{x})S_0(\psi)\Big]y,y\Big\rangle\Big|_{x_1=0}$$
$$\equiv -\Big\langle\Big[\sum_{j=2}^{k}\frac{\partial S_0(\psi)}{\partial x_j}f_j(\tilde{x})+S_0(\psi)A(\tilde{x})+A^T(\tilde{x})S_0(\psi)\Big]y,y\Big\rangle\Big|_{x_1=\pi}$$
$$\geq \varepsilon||y||^2.$$

Let us take note of the fact that because of the form of the matrix (11) matrices $A(\tilde{x})$ are also continuous with respect to $\tilde{x}$ variables, thus a smaller number of variables than x, for which the inequality

$$\big\langle[S_0(\psi)A(\tilde{x})+A^T(\tilde{x})S_0(\psi)]y,y\big\rangle \geq \varepsilon||y||^2, \qquad \varepsilon=\text{const}>0$$

holds, does not exist. Let $2n\times 2n$-dimensional matrices $B(x), M(x)\in C^0(T_k)$ have the following forms:

$$B(x)=\begin{bmatrix} B_1(x) & B_{12}(x)\\ B_{12}^T(x) & B_2(x)\end{bmatrix}, \qquad M(x)=\begin{bmatrix} 0 & M(x)\\ -M^T(x) & 0\end{bmatrix}. \tag{26}$$

THEOREM 2
Let $B(x), M(x)\in C^0(T_k)$ be of the form (26) *and satisfy conditions* (8) *and* (9)*. Then the system of equations*

$$\begin{cases} \dfrac{dy_1}{dt} = \big[-SB_1\sin\psi+\big[B_{12}^T(x)-M^T(x)\big]\,y_1\big]+\\ \qquad +\Big[-S\,[B_{12}(x)+M(x)]\sin\psi\\ \qquad +\Big(B_2(x)-0.5S\Big(\displaystyle\sum_{j=1}^{k}f_j(x)\Big)\cos\psi\Big)\Big]y_2\,,\\ \dfrac{dy_2}{dt} = B_1(x)y_1+[B_{12}(x)+M(x)]\,y_2\,,\\ \dfrac{dx}{dt} = f(x),\\ \psi = \displaystyle\sum_{j=1}^{k}x_j \end{cases} \tag{27}$$

is regular for any vector function $f(x) \in C_{Lip}(T_k)$ and any symmetric constant matrix S.

Proof. Let us consider the square forms

$$V = 2\langle y_1, y_2\rangle + \langle Sy_2, y_2\rangle \sin\psi, \qquad y_1, y_2 \in \mathbb{R}^n,\ \psi = x_1 + \ldots + x_k\,, \tag{28}$$

where the matrix S is $n\times n$-dimensional, constant and symmetric.

The matrix

$$S_0(x) = \begin{bmatrix} 0 & I_n \\ I_n & S\sin\psi \end{bmatrix}, \qquad \psi = \sum_{j=1}^{k} x_j\,,$$

which corresponds to the form (28), is non-degenerate. Obviously, the inverse matrix has the form

$$S_0^{-1}(x) = \begin{bmatrix} -S\sin\psi & I_n \\ I_n & 0 \end{bmatrix}.$$

We determine the matrix (7), when $b(x) = f(x)$:

$$S_0^{-1}(x)\left[B(x) + M(x) - 0.5\frac{\partial S_0(x)}{\partial x} f(x)\right]$$

$$= \left[\begin{matrix} -S\sin\psi + (B_{12}^T - M^T) & -S(B_{12}+M)\sin\psi + \left(B_2 - 0.5S\left(\sum_{j=1}^{k} f_j\right)\cos\psi\right) \\ B_1 & B_{12}+M \end{matrix}\right].$$

Therefore the derivative of the square form (28) along the solutions of the system (27) is positive definite. Thus the system (27) is regular (cf. [3]).

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